

1997 AIME

Time limit: 180 minutes

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1. How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?
2. The nine horizontal and nine vertical lines on an 8×8 checkerboard form r rectangles, of which s are squares. The number $\frac{s}{r}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
3. Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?

4. Circles of radii 5, 5, 8, and $\frac{m}{n}$ are mutually externally tangent, where m and n are relatively prime positive integers. Find $m + n$.
5. The number r can be expressed as a four-place decimal $0.abcd$, where a, b, c , and d represent digits, any of which could be zero. It is desired to approximate r by a fraction whose numerator is 1 or 2 and whose denominator is an integer. The closest such fraction to r is $\frac{2}{7}$. What is the number of possible values for r ?
6. Point B is in the exterior of the regular n -sided polygon $A_1A_2 \cdots A_n$, and A_1A_2B is an equilateral triangle. What is the largest value of n for which A_n, A_1 , and B are consecutive vertices of a regular polygon?
7. A car travels due east at $\frac{2}{3}$ mile per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{1}{2}\sqrt{2}$ mile per minute. At time $t = 0$, the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $\frac{1}{2}(t_1 + t_2)$.

8. How many different 4×4 arrays whose entries are all 1's and -1 's have the property that the sum of the entries in each row is 0 and the sum of the entries in each column is 0?
9. Given a nonnegative real number x , let $\langle x \rangle$ denote the fractional part of x ; that is, $\langle x \rangle = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x . Suppose that a is positive, $\langle a^{-1} \rangle = \langle a^2 \rangle$, and $2 < a^2 < 3$. Find the value of $a^{12} - 144a^{-1}$.
10. Every card in a deck has a picture of one shape — circle, square, or triangle, which is painted in one of the three colors — red, blue, or green. Furthermore, each color is applied in one of three shades — light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:
- Either each of the three cards has a different shape or all three of the cards have the same shape.
 - Either each of the three cards has a different color or all three of the cards have the same color.
 - Either each of the three cards has a different shade or all three of the cards have the same shade.

How many different complementary three-card sets are there?

11. Let

$$x = \frac{\sum_{n=1}^{44} \cos n^\circ}{\sum_{n=1}^{44} \sin n^\circ}.$$

What is the greatest integer that does not exceed $100x$?

12. The function f defined by $f(x) = \frac{ax+b}{cx+d}$, where a, b, c , and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values except $-\frac{d}{c}$. Find the unique number that is not in the range of f .

13. Let S be the set of points in the Cartesian plane that satisfy

$$\left| \left| |x| - 2 \right| - 1 \right| + \left| \left| |y| - 2 \right| - 1 \right| = 1.$$

If a model of S were built from wire of negligible thickness, then the total length of wire required would be $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a + b$.

14. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2 + \sqrt{3}} \leq |v + w|$, where m and n are relatively prime positive integers. Find $m + n$.
15. The sides of rectangle $ABCD$ have lengths 10 and 11. An equilateral triangle is drawn so that no point of the triangle lies outside $ABCD$. The maximum possible area of such a triangle can be written in the form $p\sqrt{q} - r$, where p , q , and r are positive integers, and q is not divisible by the square of any prime number. Find $p + q + r$.

Solutions: <https://live.poshenloh.com/past-contests/aime/1997/solutions>

