

1998 AIME

Time limit: 180 minutes

Typeset by: LIVE by Po-Shen Loh

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1. For how many values of k is 12^{12} the least common multiple of the positive integers 6^6 , 8^8 , and k ?
2. Find the number of ordered pairs (x, y) of positive integers that satisfy $x \leq 2y \leq 60$ and $y \leq 2x \leq 60$.
3. The graph of $y^2 + 2xy + 40|x| = 400$ partitions the plane into several regions. What is the area of the bounded region?

4. Nine tiles are numbered $1, 2, 3, \dots, 9$, respectively. Each of three players randomly selects and keeps three of the tiles, and sums those three values. The probability that all three players obtain an odd sum is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
5. Given that $A_k = \frac{k(k-1)}{2} \cos \frac{k(k-1)\pi}{2}$, find $|A_{19} + A_{20} + \dots + A_{98}|$.
6. Let $ABCD$ be a parallelogram. Extend $\overline{DA}$ through A to a point P , and let $\overline{PC}$ meet $\overline{AB}$ at Q and $\overline{DB}$ at R . Given that $PQ = 735$ and $QR = 112$, find RC .
7. Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$. Find $\frac{n}{100}$.

8. Except for the first two terms, each term of the sequence $1000, x, 1000 - x, \dots$ is obtained by subtracting the preceding term from the one before that. The last term of the sequence is the first negative term encountered. What positive integer x produces a sequence of maximum length?
9. Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where a, b , and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.
10. Eight spheres of radius 100 are placed on a flat surface so that each sphere is tangent to two others and their centers are the vertices of a regular octagon. A ninth sphere is placed on the flat surface so that it is tangent to each of the other eight spheres. The radius of this last sphere is $a + b\sqrt{c}$, where a, b , and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

11. Three of the edges of a cube are $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$, and $\overline{AD}$ is an interior diagonal. Points P , Q , and R are on $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$, respectively, so that $AP = 5$, $PB = 15$, $BQ = 15$, and $CR = 10$. What is the area of the polygon that is the intersection of plane PQR and the cube?
12. Let ABC be equilateral, and D , E , and F be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. There exist points P , Q , and R on $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$, respectively, with the property that P is on $\overline{CQ}$, Q is on $\overline{AR}$, and R is on $\overline{BP}$. The ratio of the area of triangle ABC to the area of triangle PQR is $a + b\sqrt{c}$, where a , b , and c are integers, and c is not divisible by the square of any prime. What is $a^2 + b^2 + c^2$?
13. If $\{a_1, a_2, a_3, \dots, a_n\}$ is a set of real numbers, indexed so that $a_1 < a_2 < a_3 < \dots < a_n$, its complex power sum is defined to be $a_1i + a_2i^2 + a_3i^3 + \dots + a_ni^n$, where $i^2 = -1$. Let S_n be the sum of the complex power sums of all nonempty subsets of $\{1, 2, \dots, n\}$. Given that $S_8 = -176 - 64i$ and $S_9 = p + qi$, where p and q are integers, find $|p| + |q|$.
14. An $m \times n \times p$ rectangular box has half the volume of an $(m + 2) \times (n + 2) \times (p + 2)$ rectangular box, where m , n , and p are integers, and $m \leq n \leq p$. What is the largest possible value of p ?

15. Define a domino to be an ordered pair of distinct positive integers. A proper sequence of dominos is a list of distinct dominos in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not both appear for any i and j . Let D_{40} be the set of all dominos whose coordinates are no larger than 40. Find the length of the longest proper sequence of dominos that can be formed using the dominos of D_{40} .

Solutions: <https://live.poshenloh.com/past-contests/aime/1998/solutions>

