1998 AIME Solutions

Typeset by: LIVE by Po-Shen Loh
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1. For how many values of k is 12^{12} the least common multiple of the positive integers 6^6, 8^8, and k?

Solution:

Since $12^{12} = 2^{24}3^{12}$, $6^6 = 2^6 3^6$, and $8^8 = 2^{24}$, the number k can involve no primes other than 2 and 3, so write $k = 2^a 3^b$. The least common multiple of the three numbers is then $2^{\max(24,\, a)}\, 3^{\max(6,\, b)}$.

Matching this to $2^{24}3^{12}$ requires $\max(24, a) = 24$, i.e. $0 \le a \le 24$, and $\max(6, b) = 12$, i.e. $b = 12$. That gives 25 choices for a and one for b, so there are 25 values of k.

2. Find the number of ordered pairs (x, y) of positive integers that satisfy $x \leq 2y \leq 60$ and $y \leq 2x \leq 60$.

Solution:

The chains unpack into four conditions: $x \leq 2y, 2y \leq 60, y \leq 2x$, and $2x \leq 60$. So (x, y) lies in the square $1 \leq x, y \leq 30$, and within it we must avoid $x > 2y$ and $y > 2x$, which cannot both happen.

Pairs with $y > 2x$: for each x from 1 to 14 the values $y = 2x + 1, \ldots, 30$ work, giving $\sum_{x=1}^{14}(30 - 2x) = 420 - 210 = 210$ pairs. By the symmetry swapping x and y, there are also 210 pairs with $x > 2y$.

The answer is $30 \cdot 30 - 210 - 210 = 480$.

3. The graph of $y^2 + 2xy + 40|x| = 400$ partitions the plane into several regions. What is the area of the bounded region?

Solution:

For $x \geq 0$ rewrite the equation as $2x(y + 20) = 400 - y^2 = (20 - y)(20 + y)$, so either $y = -20$ or $y = 20 - 2x$. For $x \leq 0$ it becomes $2x(y - 20) = -(y - 20)(y + 20)$, so either $y = 20$ or $y = -20 - 2x$. The graph therefore consists of two horizontal rays and two rays of slope -2.

These rays bound a parallelogram: the top edge runs from $(-20, 20)$ to $(0, 20)$ along $y = 20$, the bottom edge from $(0, -20)$ to $(20, -20)$ along $y = -20$, and the two slanted edges of slope -2 connect them.

The parallelogram has horizontal base 20 and height 40 between the lines $y = 20$ and $y = -20$, so its area is $20 \cdot 40 = 800$.

4. Nine tiles are numbered $1, 2, 3, \ldots, 9,$ respectively. Each of three players randomly selects and keeps three of the tiles, and sums those three values. The probability that all three players obtain an odd sum is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

A player's three tiles have an odd sum exactly when the player holds an odd number of odd tiles — one or three. The nine tiles include five odd and four even, and the only way to split five odd tiles into three groups of size one or three is $3 + 1 + 1$.

Count favorable deals: choose which player gets three odd tiles (3 ways), choose that player's odd tiles ($\binom{5}{3} = 10$ ways), give one of the two remaining odd tiles to each other player (2 ways), then split the four even tiles two and two between those players ($\binom{4}{2} = 6$ ways), for $3 \cdot 10 \cdot 2 \cdot 6 = 360$ deals. The total number of deals is $\binom{9}{3}\binom{6}{3} = 84 \cdot 20 = 1680.$

The probability is $\frac{360}{1680} = \frac{3}{14}$, so $m + n = 3 + 14 = 17.$

5. Given that $A_k = \frac{k(k-1)}{2} \cos \frac{k(k-1)\pi}{2}$, find $|A_{19} + A_{20} + \cdots + A_{98}|$.

Solution:

Since $k(k-1)$ is even, $n_k = \frac{k(k-1)}{2}$ is an integer and $\cos \frac{k(k-1)\pi}{2} = \cos(n_k \pi) = (-1)^{n_k}$. The parity of the triangular number n_k depends only on $k \bmod 4$: it is even for $k \equiv 0, 1 \pmod 4$ and odd for $k \equiv 2, 3 \pmod 4$. So $A_k = n_k$ when $k \equiv 0, 1 \pmod 4$ and $A_k = -n_k$ when $k \equiv 2, 3 \pmod 4$.

Group the 80 terms into 20 consecutive blocks of four starting at $k = 19 \equiv 3 \pmod 4$. Using $n_{j+1} - n_j = j$, each block with $k \equiv 3 \pmod 4$ collapses:

$$\begin{aligned} &A_k + A_{k+1} \\ &\quad + A_{k+2} + A_{k+3} \\ &= (n_{k+1} - n_k) \\ &\quad - (n_{k+3} - n_{k+2}) \\ &= k - (k+2) \\ &= -2. \end{aligned}$$

The total is $20 \cdot (-2) = -40$, so the requested absolute value is 40.

6. Let $ABCD$ be a parallelogram. Extend $\overline{DA}$ through A to a point P, and let $\overline{PC}$ meet $\overline{AB}$ at Q and $\overline{DB}$ at R. Given that $PQ = 735$ and $QR = 112$, find RC.

Solution:

Let $a = \frac{PA}{AD}$. Since $AQ \parallel DC$, triangles PAQ and PDC are similar, so $\frac{PQ}{PC} = \frac{PA}{PD} = \frac{a}{a+1}$. Since $BC \parallel AD$, i.e. $BC \parallel PD$, triangles RBC and RDP are similar, so $\frac{RC}{RP} = \frac{BC}{PD} = \frac{1}{a+1}$, which gives $\frac{RC}{PC} = \frac{1}{a+2}$.

Writing $PC = L$, we get $PQ = \frac{a}{a+1}L$, $RC = \frac{L}{a+2}$, and

$$\begin{aligned} QR &= L - PQ - RC \\ &= \frac{L}{(a+1)(a+2)}. \end{aligned}$$

Hence $\frac{PQ}{QR} = a(a+2) = \frac{735}{112} = \frac{105}{16}$, so $16a^2 + 32a - 105 = 0$, which factors as $(4a-7)(4a+15) = 0$, giving $a = \frac{7}{4}$.

Finally $RC = (a+1)\, QR = \frac{11}{4} \cdot 112 = 308.$

7. Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$. Find $\frac{n}{100}$.

Solution:

Write $x_i = 2y_i - 1$ where each y_i is a positive integer. Then $x_1 + x_2 + x_3 + x_4 = 98$ becomes $2(y_1 + y_2 + y_3 + y_4) - 4 = 98$, so $y_1 + y_2 + y_3 + y_4 = 51$.

By stars and bars, the number of solutions in positive integers is $\binom{50}{3} = 19600.$ Therefore $\frac{n}{100} = 196.$

8. Except for the first two terms, each term of the sequence $1000, x, 1000 - x, \ldots$ is obtained by subtracting the preceding term from the one before that. The last term of the sequence is the first negative term encountered. What positive integer x produces a sequence of maximum length?

Solution:

Computing terms, $a_3 = 1000 - x, a_4 = 2x - 1000, a_5 = 2000 - 3x, a_6 = 5x - 3000,$ and in general

$$a_{2k+1} = 1000F_{2k-1} - xF_{2k},$$
$$a_{2k+2} = xF_{2k+1} - 1000F_{2k},$$

where $F_1 = F_2 = 1, F_3 = 2, \ldots$ are the Fibonacci numbers. The sequence keeps going exactly as long as its terms stay nonnegative, so a long sequence requires $\frac{x}{1000}$ to be squeezed between the ratios $\frac{F_{2k}}{F_{2k+1}}$ and $\frac{F_{2k-1}}{F_{2k}}$ for larger and larger k.

For the first 13 terms to be nonnegative we need $a_{12} = 89x - 55000 \geq 0$ and $a_{13} = 89000 - 144x \geq 0$, i.e. $617.9\ldots \leq x \leq 618.05\ldots$, so $x = 618$. If $x \leq 617$ the sequence turns negative by a_{12}, and if $x \geq 619$ it turns negative by a_{13}, so every other integer gives a shorter sequence.

Indeed $x = 618$ yields $1000, 618, 382, 236, 146, 90, 56, 34, 22, 12, 10, 2, 8, -6$, a sequence of 14 terms, the maximum possible. The answer is 618.

9. Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where $a, b,$ and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

Solution:

Let the arrival times be x and y minutes after 9 a.m., so (x, y) is uniform in a 60×60 square. The two people meet exactly when $|x - y| < m$.

The non-meeting region $|x - y| \geq m$ consists of two right triangles with legs $60 - m$, with total area $(60 - m)^2$. Meeting with probability 40% means

$$(60 - m)^2 = 0.6 \cdot 3600 = 2160,$$

so $60 - m = \sqrt{2160} = 12\sqrt{15}$.

Thus $m = 60 - 12\sqrt{15}$, and $a + b + c = 60 + 12 + 15 = 87$.

10. Eight spheres of radius 100 are placed on a flat surface so that each sphere is tangent to two others and their centers are the vertices of a regular octagon. A ninth sphere is placed on the flat surface so that it is tangent to each of the other eight spheres. The radius of this last sphere is $a + b\sqrt{c}$, where $a, b,$ and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

Solution:

The eight centers are at height 100, at the vertices of a regular octagon of side 200 (adjacent spheres are tangent). If the ninth sphere has radius r, it rests on the surface with its center at height r directly above the octagon's center, and tangency to each sphere gives $\sqrt{R^2 + (r-100)^2} = r + 100$, where R is the octagon's circumradius. Hence

$$\begin{aligned} R^2 &= (r+100)^2 - (r-100)^2 \\ &= 400r. \end{aligned}$$

A side of a regular octagon subtends 45° at the center, so $200 = 2R\sin 22.5^\circ$ and, using $\sin^2 22.5^\circ = \frac{1-\cos 45^\circ}{2} = \frac{2-\sqrt{2}}{4}$,

$$\begin{aligned} R^2 &= \frac{10000}{\sin^2 22.5^\circ} \\ &= \frac{40000}{2-\sqrt{2}} \\ &= 20000\,(2+\sqrt{2}). \end{aligned}$$

Then $r = \frac{R^2}{400} = 50\,(2+\sqrt{2}) = 100 + 50\sqrt{2}$, so $a + b + c = 100 + 50 + 2 = 152.$

11. Three of the edges of a cube are $\overline{AB}, \overline{BC}$, and $\overline{CD}$, and $\overline{AD}$ is an interior diagonal. Points P, Q, and R are on $\overline{AB}, \overline{BC}$, and $\overline{CD}$, respectively, so that $AP = 5, PB = 15, BQ = 15$, and $CR = 10$. What is the area of the polygon that is the intersection of plane PQR and the cube?

Solution:

The cube has side 20. Take $B = (0,0,0), A = (20,0,0), C = (0,20,0)$, and $D = (0,20,20)$, so $\overline{AD}$ is an interior diagonal. Then $P = (15,0,0), Q = (0,15,0), R = (0,20,10)$, and the plane through them is $2x + 2y - z = 30$.

Evaluating $2x + 2y - z$ at the cube's vertices and checking all twelve edges, the plane also crosses the edges at $(5,20,20), (20,5,20)$, and $(20,0,10)$, so the cross-section is the hexagon with vertices $(15,0,0), (0,15,0), (0,20,10), (5,20,20)$, $(20,5,20), (20,0,10)$ in order. Its projection onto the xy-plane is the hexagon $(15,0), (0,15), (0,20), (5,20), (20,5), (20,0)$, whose area by the shoelace formula is 175.

The plane's unit normal $\frac{1}{3}(2,2,-1)$ has vertical component of magnitude $\frac{1}{3}$, so projecting onto the xy-plane multiplies area by $\frac{1}{3}$. The cross-section therefore has area $3 \cdot 175 = 525$.

12. Let ABC be equilateral, and $D, E,$ and F be the midpoints of $\overline{BC}, \overline{CA},$ and $\overline{AB}$, respectively. There exist points $P, Q,$ and R on $\overline{DE}, \overline{EF},$ and $\overline{FD}$, respectively, with the property that P is on $\overline{CQ}$, Q is on $\overline{AR}$, and R is on $\overline{BP}$. The ratio of the area of triangle ABC to the area of triangle PQR is $a + b\sqrt{c}$, where $a, b,$ and c are integers, and c is not divisible by the square of any prime. What is $a^2 + b^2 + c^2$?

Solution:

Place $A = (0, \sqrt{3}), B = (-1, 0), C = (1, 0)$, so $D = (0, 0), E = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), F = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Write $P = D + x(E - D), Q = E + y(F - E)$, and $R = F + z(D - F)$, where $0 \le x, y, z \le 1$. Computing the three two-dimensional cross products, the collinearities C, P, Q; A, Q, R; and B, R, P give, respectively,

$$\begin{aligned} x(1 + y) &= 1, \\ y(1 + z) &= 1, \\ z(1 + x) &= 1. \end{aligned}$$

Let $f(u) = \frac{1}{1+u}$. The equations say $x = f(y), y = f(z)$, and $z = f(x)$, so

$$x = f(f(f(x))) = \frac{x + 2}{2x + 3}.$$

Hence $x^2 + x - 1 = 0$, and positivity gives $x = \frac{\sqrt{5}-1}{2}$. The equations then force $y = z = x$; call this common value t. Thus the desired configuration really is the symmetric one, rather than merely being assumed to be.

With $P = \left(\frac{t}{2}, \frac{t\sqrt{3}}{2}\right)$ and $Q = \left(\frac{1}{2} - t, \frac{\sqrt{3}}{2}\right)$, both triangles are equilateral with center $G = \left(0, \frac{\sqrt{3}}{3}\right)$. Therefore the area ratio is $\frac{GA^2}{GP^2}$. Using $t^2 = 1 - t$,

$$\begin{aligned}
GP^2 &= \frac{t^2}{4} + 3\left(\frac{t}{2} - \frac{1}{3}\right)^2 \\
&= t^2 - t + \frac{1}{3} \\
&= \frac{7}{3} - \sqrt{5}, \\
GA^2 &= \frac{4}{3}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{[ABC]}{[PQR]} &= \frac{\frac{4}{3}}{\frac{7}{3} - \sqrt{5}} \\
&= \frac{4}{7 - 3\sqrt{5}} \\
&= 7 + 3\sqrt{5},
\end{aligned}$$

so $a = 7, b = 3, c = 5,$ and $a^2 + b^2 + c^2 = 49 + 9 + 25 = 83.$

13. If $\{a_1, a_2, a_3, \ldots, a_n\}$ is a set of real numbers, indexed so that $a_1 < a_2 < a_3 < \cdots < a_n$, its complex power sum is defined to be $a_1 i + a_2 i^2 + a_3 i^3 + \cdots + a_n i^n$, where $i^2 = -1$. Let S_n be the sum of the complex power sums of all nonempty subsets of $\{1, 2, \ldots, n\}$. Given that $S_8 = -176 - 64i$ and $S_9 = p + qi$, where p and q are integers, find $|p| + |q|$.

Solution:

Split the nonempty subsets of $\{1, \ldots, 9\}$ by whether they contain 9. Those without 9 contribute S_8. A subset containing 9 is $T \cup \{9\}$ for a (possibly empty) $T \subseteq \{1, \ldots, 8\}$, and since 9 is its largest element, its complex power sum is the complex power sum of T plus $9i^{|T|+1}$. Summing over all T gives another S_8 plus

$$\sum_{k=0}^{8} \binom{8}{k} 9\, i^{k+1} = 9i\,(1+i)^8.$$

Since $(1+i)^2 = 2i$, we get $(1+i)^8 = (2i)^4 = 16$, so

$$\begin{aligned} S_9 &= 2S_8 + 144i \\ &= 2(-176 - 64i) + 144i \\ &= -352 + 16i. \end{aligned}$$

Therefore $|p| + |q| = 352 + 16 = 368.$

14. An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where $m, n,$ and p are integers, and $m \leq n \leq p$. What is the largest possible value of p?

Solution:

The condition $2mnp = (m+2)(n+2)(p+2)$ rewrites as

$$\left(1+\frac{2}{m}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{2}{p}\right) = 2.$$

If $m = 1$ the first factor alone is $3 > 2$, and if $m = 2$ it equals 2 while the other factors exceed 1; both are impossible. If $m \geq 5$, then since $n \geq m$ the first two factors are at most $\left(\frac{7}{5}\right)^2 = \frac{49}{25}$, forcing $1 + \frac{2}{p} \geq \frac{50}{49}$, i.e. $p \leq 98$.

For $m = 4$ the equation becomes $4np = 3(n+2)(p+2)$, i.e. $(n-6)(p-6) = 48$, so $p \leq 54$. For $m = 3$ it becomes $6np = 5(n+2)(p+2)$, i.e. $np - 10n - 10p - 20 = 0$, or $(n-10)(p-10) = 120$. Both factors must be positive (if $n, p < 10$ the product $(10-n)(10-p)$ is at most 49), so the largest p comes from $n - 10 = 1:$ $n = 11$ and $p = 130$. Indeed $2 \cdot 3 \cdot 11 \cdot 130 = 8580 = 5 \cdot 13 \cdot 132$.

Since every other case yields $p \leq 98$, the largest possible value is $p = 130$.

15. Define a domino to be an ordered pair of distinct positive integers. A proper sequence of dominos is a list of distinct dominos in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not both appear for any i and j. Let D_{40} be the set of all dominos whose coordinates are no larger than 40. Find the length of the longest proper sequence of dominos that can be formed using the dominos of D_{40}.

Solution:

A domino (i, j) is an oriented edge of the complete graph on vertices $1, \ldots, 40$, and the rule that (i, j) and (j, i) cannot both appear means each of the $\binom{40}{2} = 780$ edges is available at most once. A proper sequence is exactly a trail: a walk that repeats no edge. In any trail, every vertex other than the two endpoints is entered and left equally often, so it has even degree in the set of edges used.

In the complete graph every vertex has odd degree 39, so at least 38 vertices must have odd degree in the set of unused edges, and a graph with 38 odd-degree vertices has at least $\frac{38}{2} = 19$ edges. Hence at most $780 - 19 = 761$ dominos can be used.

Conversely, set aside the 19 disjoint edges $(3, 4), (5, 6), \ldots, (39, 40)$. The remaining graph is connected and only vertices 1 and 2 have odd degree, so it has an Euler trail traversing all 761 remaining edges; orienting each edge in the direction of travel gives a proper sequence of length 761.

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