2003 AIME II Solutions

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1. The product N of three positive integers is 6 times their sum, and one of the integers is the sum of the other two. Find the sum of all possible values of N.

Solution:

Let the integers be a, b, and $c = a + b$. Then

$$\begin{aligned} N &= abc = 6(a+b+c) \\ &= 6 \cdot 2c = 12c, \end{aligned}$$

and cancelling c from $abc = 12c$ leaves $ab = 12$.

The factorizations $(a, b) = (1, 12), (2, 6), (3, 4)$ give $c = 13, 8, 7$ and $N = 12c = 156, 96, 84$. The sum of all possible values is $156 + 96 + 84 = 336$.

2. Let N be the greatest integer multiple of $8,$ no two of whose digits are the same. What is the remainder when N is divided by $1000?$

Solution:

An integer is divisible by 8 exactly when the number formed by its last three digits is. To make N as large as possible, use all ten digits once each and put the largest digits first: the leading digits are $9876543,$ and the final three digits are some arrangement of $0, 1, 2$ — provided one of those arrangements is a multiple of $8.$

Checking $012, 021, 102, 120, 201, 210,$ the only multiple of 8 is $120.$ So $N = 9{,}876{,}543{,}120,$ and the remainder upon division by 1000 is $120.$

3. Define a *good word* as a sequence of letters that consists only of the letters $A, B,$ and C — some of these letters may not appear in the sequence — and in which A is never immediately followed by B, B is never immediately followed by $C,$ and C is never immediately followed by $A.$ How many seven-letter good words are there?

Solution:

Each letter rules out exactly one successor (A forbids B, B forbids C, C forbids A), so whatever letter has just been written, exactly 2 of the 3 letters may come next.

With 3 choices for the first letter and 2 for each of the remaining six positions, the number of seven-letter good words is $3 \cdot 2^6 = 192.$

4. In a regular tetrahedron, the centers of the four faces are the vertices of a smaller tetrahedron. The ratio of the volume of the smaller tetrahedron to that of the larger is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Use position vectors, and let $G = \frac{A+B+C+D}{4}$ be the centroid of the tetrahedron. The center of the face opposite A is

$$\begin{aligned} \frac{B+C+D}{3} &= \frac{4G-A}{3} \\ &= G - \frac{1}{3}(A-G), \end{aligned}$$

so each face center is the image of the opposite vertex under the homothety centered at G with ratio $-\frac{1}{3}$.

Hence the smaller tetrahedron is similar to the larger with ratio $\frac{1}{3}$, and its volume is $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$ of the larger. Thus $m+n = 1+27 = 28.$

5. A cylindrical log has diameter 12 inches. A wedge is cut from the log by making two planar cuts that go entirely through the log. The first is perpendicular to the axis of the cylinder, and the plane of the second cut forms a 45° angle with the plane of the first cut. The intersection of these two planes has exactly one point in common with the log. The number of cubic inches in the wedge can be expressed as $n\pi$, where n is a positive integer. Find n.

Solution:

Take the first cut as horizontal. The line where the two cutting planes meet touches the log at exactly one point, so it is tangent to the circular cross-section of radius 6. The wedge therefore stands over the entire disk: its height is 0 at the tangent point and, because the second cut is at 45°, it rises linearly to 12 at the diametrically opposite point.

Pair each point of the disk with its mirror image through the center: the wedge's heights over the two points add to exactly 12. So two copies of the wedge fit together into a cylinder of radius 6 and height 12, and the wedge's volume is

$$\frac{1}{2}\,\pi \cdot 6^2 \cdot 12 = 216\pi.$$

Thus $n = 216$.

6. In $\triangle ABC, AB = 13, BC = 14, AC = 15,$ and point G is the intersection of the medians. Points A', B', and C' are the images of $A, B,$ and $C,$ respectively, after a 180° rotation about $G.$ What is the area of the union of the two regions enclosed by the triangles ABC and $A'B'C'$?

Solution:

A 180° rotation takes each line to a parallel line, so $\triangle A'B'C'$ is congruent to $\triangle ABC$ with parallel sides. View BC as horizontal and let h be the height of A above it. The centroid G is at height $\frac{h}{3}$, so A', the reflection of A through $G,$ is at height $2 \cdot \frac{h}{3} - h = -\frac{h}{3}$, on the far side of line BC, while B' and C' are at height $\frac{2h}{3}.$

Line BC therefore slices off the corner of $\triangle A'B'C'$ at A' : the cut is parallel to $B'C'$, and the corner's height $\frac{h}{3}$ is one third of the triangle's full height $h,$ so the corner is similar with ratio $\frac{1}{3}$ and has area $\frac{1}{9}[ABC].$ The same happens at each side of $\triangle ABC,$ and these three corners are exactly the part of $\triangle A'B'C'$ outside $\triangle ABC.$ Hence the union has area

$$\begin{aligned}&[ABC] + 3 \cdot \tfrac{1}{9}[ABC]\\&= \tfrac{4}{3}[ABC].\end{aligned}$$

By Heron's formula with $s = 21, [ABC] = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84,$ so the union has area $\frac{4}{3} \cdot 84 = 112.$

7. Find the area of rhombus $ABCD$ given that the radii of the circles circumscribed around triangles ABD and ACD are 12.5 and 25, respectively.

Solution:

Let s be the side length and $\alpha = \angle BAC$ (the diagonal AC bisects angle A). The diagonals then have lengths $AC = 2s\cos\alpha$ and $BD = 2s\sin\alpha$. In triangle ABD, side BD subtends the angle $\angle BAD = 2\alpha$, so the extended law of sines gives

$$\begin{aligned} 12.5 = R_1 &= \frac{BD}{2\sin 2\alpha} \\ &= \frac{2s\sin\alpha}{4\sin\alpha\cos\alpha} \\ &= \frac{s}{2\cos\alpha}. \end{aligned}$$

In triangle ACD, side AC subtends $\angle ADC = 180^\circ - 2\alpha$, so similarly $25 = R_2 = \frac{s}{2\sin\alpha}$.

Dividing, $\tan\alpha = \frac{R_1}{R_2} = \frac{1}{2}$, so $\sin\alpha = \frac{1}{\sqrt{5}}$ and $\cos\alpha = \frac{2}{\sqrt{5}}$. Then $s = 2R_2\sin\alpha = \frac{50}{\sqrt{5}} = 10\sqrt{5}$.

The area is half the product of the diagonals:

$$\begin{aligned} &\frac{1}{2}\cdot 2s\cos\alpha\cdot 2s\sin\alpha \\ &= 2s^2\sin\alpha\cos\alpha \\ &= 2\cdot 500\cdot\frac{2}{5} = 400. \end{aligned}$$

8. Find the eighth term of the sequence $1440, 1716, 1848, \ldots,$ whose terms are formed by multiplying the corresponding terms of two arithmetic sequences.

Solution:

The nth term of an arithmetic sequence is linear in n, so the product of corresponding terms of two arithmetic sequences is a quadratic $t_n = an^2 + bn + c$. Indexing the given terms by $n = 0, 1, 2:$

$$\begin{aligned} c &= 1440, \\ a + b + c &= 1716, \\ 4a + 2b + c &= 1848, \end{aligned}$$

which give $a + b = 276$ and $2a + b = 204$, so $a = -72, b = 348, c = 1440.$

The eighth term is $t_7 = -72 \cdot 49 + 348 \cdot 7 + 1440 = 348.$ (Indeed $t_n = (180 - 24n)(8 + 3n)$, a product of two arithmetic sequences matching the given terms.)

9. Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that $z_1, z_2, z_3,$ and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

Solution:

Polynomial division gives

$$P(x) = Q(x)\,(x^2 + 1) \\ + x^2 - x + 1,$$

so $P(z_i) = z_i^2 - z_i + 1$ for each root z_i of Q.

By Vieta's formulas for $Q(x) = x^4 - x^3 - x^2 - 1$, we have $\sum z_i = 1$ and $\sum_{i<j} z_i z_j = -1$, so $\sum z_i^2 = \left(\sum z_i\right)^2 - 2\sum_{i<j} z_i z_j = 1 + 2 = 3$. Therefore

$$\sum_{i=1}^{4} P(z_i) = 3 - 1 + 4 = 6.$$

10. Two positive integers differ by 60. The sum of their square roots is the square root of an integer that is not a perfect square. What is the maximum possible sum of the two integers?

Solution:

Let the integers be x and $x+60$, and suppose $\sqrt{x}+\sqrt{x+60}=\sqrt{y}$. Squaring, $y=2x+60+2\sqrt{x(x+60)}$, so $x(x+60)$ must be a perfect square, say z^2. Completing the square,

$$(x+30)^2-z^2=900,$$

i.e.

$$(x+30+z)(x+30-z)=900.$$

The two factors have the same parity and their product is even, so both are even.

The factor pairs $(450,2),(150,6),(90,10),(50,18)$ give $x+30=226,78,50,34$, so $x=196,48,20,4$. For $x=196$ the integers are 196 and 256, both perfect squares, so $\sqrt{y}=14+16=30$ and $y=900$ is a perfect square — not allowed. For $x=48$ the integers are 48 and 108, with $\sqrt{48}+\sqrt{108}=4\sqrt{3}+6\sqrt{3}=\sqrt{300}$, and 300 is not a perfect square.

The maximum possible sum is therefore $48+108=156$.

11. Triangle ABC is a right triangle with $AC = 7, BC = 24$, and right angle at C. Point M is the midpoint of $\overline{AB}$, and D is on the same side of line AB as C so that $AD = BD = 15$. Given that the area of $\triangle CDM$ can be expressed as $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

Solution:

The hypotenuse is $AB = \sqrt{7^2 + 24^2} = 25$, and the median to the hypotenuse gives $CM = \frac{25}{2}$. Since $AD = BD$, point D lies on the perpendicular to AB at M, so $DM \perp AB$ and

$$\begin{aligned} DM &= \sqrt{15^2 - \left(\tfrac{25}{2}\right)^2} \\ &= \sqrt{\tfrac{275}{4}} = \tfrac{5\sqrt{11}}{2}. \end{aligned}$$

Let $\beta = \angle AMC$. In triangle AMC with $AM = CM = \frac{25}{2}$ and $AC = 7$, the law of cosines gives

$$\begin{aligned} \cos\beta &= \frac{\left(\tfrac{25}{2}\right)^2 + \left(\tfrac{25}{2}\right)^2 - 7^2}{2 \cdot \tfrac{25}{2} \cdot \tfrac{25}{2}} \\ &= \frac{527}{625}. \end{aligned}$$

Since C and D are on the same side of AB and $MD \perp AB$, we have $\angle CMD = 90^\circ - \beta$, so $\sin\angle CMD = \cos\beta$.

Therefore

$$\begin{aligned} [CDM] &= \frac{1}{2} \cdot \frac{25}{2} \cdot \frac{5\sqrt{11}}{2} \cdot \frac{527}{625} \\ &= \frac{527\sqrt{11}}{40}, \end{aligned}$$

and $m + n + p = 527 + 11 + 40 = 578$.

12. The members of a distinguished committee were choosing a president, and each member gave one vote to one of the 27 candidates. For each candidate, the exact percentage of votes the candidate got was smaller by at least 1 than the number of votes for that candidate. What is the smallest possible number of members of the committee?

Solution:

Let t be the number of members. A candidate with n votes has percentage $\frac{100n}{t}$, so the condition is $\frac{100n}{t} \leq n - 1$, which rearranges to $n(t - 100) \geq t$. This forces $t > 100$ and

$$n \geq \frac{t}{t - 100}.$$

If $t \leq 133$, then $\frac{t}{t-100} \geq \frac{133}{33} > 4$, so every candidate needs at least 5 votes, and the total is at least $27 \cdot 5 = 135 > t$ — impossible.

For $t = 134$, each candidate needs $n \geq \frac{134}{34}$, i.e. at least 4 votes, and this is achievable: let 26 candidates receive 4 votes each and one receive 30. Indeed $\frac{400}{134} \approx 2.99 \leq 3$ and $\frac{3000}{134} \approx 22.4 \leq 29$. So the smallest possible number of members is 134.

13. A bug starts at a vertex of an equilateral triangle. On each move, it randomly selects one of the two vertices where it is not currently located, and crawls along a side of the triangle to that vertex. Given that the probability that the bug moves to its starting vertex on its tenth move is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Solution:

Let p_n be the probability that the bug is at its starting vertex after n moves, so $p_0 = 1$. The bug is home after move $n+1$ exactly when it was elsewhere after move n (probability $1-p_n$) and then chose the starting vertex (probability $\frac{1}{2}$):

$$p_{n+1} = \frac{1}{2}(1-p_n).$$

The fixed point of this recurrence is $\frac{1}{3}$, and $p_{n+1} - \frac{1}{3} = -\frac{1}{2}\left(p_n - \frac{1}{3}\right)$, so $p_n = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$.

For $n = 10$:

$$\begin{aligned} p_{10} &= \frac{1}{3}\left(1 + \frac{2}{1024}\right) \\ &= \frac{1}{3} \cdot \frac{1026}{1024} = \frac{171}{512}. \end{aligned}$$

Since $171 = 9 \cdot 19$ and $512 = 2^9$ share no factor, $m+n = 171 + 512 = 683.$

14. Let $A=(0,0)$ and $B=(b,2)$ be points on the coordinate plane. Let $ABCDEF$ be a convex equilateral hexagon such that $\angle FAB=120^\circ$, $\overline{AB}\parallel\overline{DE}$, $\overline{BC}\parallel\overline{EF}$, $\overline{CD}\parallel\overline{FA}$, and the y-coordinates of its vertices are distinct elements of the set $\{0,2,4,6,8,10\}$. The area of the hexagon can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

Solution:

Opposite sides are parallel, equal in length, and traversed in opposite directions, so $\overrightarrow{AB}=\overrightarrow{ED}, \overrightarrow{BC}=\overrightarrow{FE}, \overrightarrow{CD}=\overrightarrow{AF}$: the hexagon is centrally symmetric, and opposite vertices' y-coordinates share a common sum, namely $\frac{0+2+\cdots+10}{3}=10$. From $y_A=0$ and $y_B=2$ we get $y_D=10, y_E=8$, and convexity puts $y_C=6, y_F=4$. Write $\overrightarrow{AB}=(b,2), \overrightarrow{BC}=(p,4), \overrightarrow{CD}=(q,4)$. Equal side lengths give $s^2=b^2+4=p^2+16=q^2+16$, so $p=\pm q$; since $p=q$ would make B,C,D collinear, $p=-q$.

Since $\overrightarrow{AF}=\overrightarrow{CD}$, we have $F=(q,4)$, and $\angle FAB=120^\circ$ gives

$$\begin{aligned}\overrightarrow{AB}\cdot\overrightarrow{AF}&=bq+8\\&=-\frac{s^2}{2}=-\frac{b^2+4}{2}.\end{aligned}$$

Taking $b>0$ forces $q<0$, so $q=-\sqrt{b^2-12}$, and the equation becomes $b\sqrt{b^2-12}=\frac{b^2+20}{2}$. Squaring yields $3b^4-88b^2-400=0$, so $b^2=\frac{100}{3}$, giving $b=\frac{10}{\sqrt{3}}, q=-\frac{8}{\sqrt{3}}, p=\frac{8}{\sqrt{3}}$.

The vertices are $A=(0,0), B=\left(\frac{10}{\sqrt{3}},2\right), C=(6\sqrt{3},6), D=\left(\frac{10}{\sqrt{3}},10\right), E=(0,8), F=\left(-\frac{8}{\sqrt{3}},4\right)$. The hexagon splits into the parallelogram $ABDE$, with vertical side $AE=8$ and horizontal offset b (area $8b$), plus the two congruent triangles

BCD and EFA, each with vertical base 8 and horizontal height $\frac{8}{\sqrt{3}}$. The total area is

$$\begin{aligned} &8 \cdot \frac{10}{\sqrt{3}} + 2 \cdot \frac{1}{2} \cdot 8 \cdot \frac{8}{\sqrt{3}} \\ &= \frac{144}{\sqrt{3}} = 48\sqrt{3}, \end{aligned}$$

so $m + n = 48 + 3 = 51.$

15. Let

$$P(x) = 24x^{24} + \sum_{j=1}^{23} (24-j)\left(x^{24-j} + x^{24+j}\right).$$

Let $z_1, z_2, \ldots, z_r$ be the distinct zeros of $P(x)$, and let $z_k^2 = a_k + b_k i$ for $k = 1, 2, \ldots, r$, where $i = \sqrt{-1}$, and a_k and b_k are real numbers. Let

$$\sum_{k=1}^{r} |b_k| = m + n\sqrt{p},$$

where $m, n,$ and p are integers and p is not divisible by the square of any prime. Find $m + n + p$.

Solution:

The coefficient of x^k in $P(x)$ is $24 - |24-k|$ for $1 \le k \le 47$, and consecutive coefficients differ by $+1$ up through x^{24} and by -1 afterwards. Multiplying by $1-x$ therefore telescopes:

$$\begin{aligned}(1-x)&P(x)\\ &= (x + x^2 + \cdots + x^{24})\\ &\quad - (x^{25} + \cdots + x^{48})\\ &= (x + x^2 + \cdots + x^{24})\\ &\quad \cdot (1 - x^{24}),\end{aligned}$$

so for $x \neq 1$,

$$P(x) = x\left(\frac{x^{24}-1}{x-1}\right)^2.$$

The distinct zeros of P are therefore 0 together with the 24th roots of unity other than $1: z_k = \cos 15k^\circ + i \sin 15k^\circ$ for $k = 1, \ldots, 23$. The zero 0 contributes nothing, and

$z_k^2 = \cos 30k^\circ + i\sin 30k^\circ$, so $|b_k| = |\sin 30k^\circ|$.

As k runs from 1 to 12, the values $|\sin 30k^\circ|$ are $\frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0$ repeated twice, summing to $4 + 2\sqrt{3}$; the terms for $k = 13, \ldots, 23$ repeat those for $k = 1, \ldots, 11$ and add another $4 + 2\sqrt{3}$. The total is $8 + 4\sqrt{3}$, so $m + n + p = 8 + 4 + 3 = 15$.

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