# 2004 AIME I Solutions

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**1.** The digits of a positive integer $n$ are four consecutive integers in decreasing order when read from left to right. What is the sum of the possible remainders when $n$ is divided by $37$?

## Solution:

If the leading digit is $a$, the digits are $a, a-1, a-2, a-3$ with $a = 3, 4, \ldots, 9$, so

$$\begin{aligned} n &= 1000a + 100(a-1) \\ &\quad + 10(a-2) + (a-3) \\ &= 1111a - 123. \end{aligned}$$

Since $1111 = 30 \cdot 37 + 1$ and $123 = 3 \cdot 37 + 12$, we get $n \equiv a - 12 \equiv a + 25 \pmod{37}$. For $a = 3, \ldots, 9$ the values $a + 25$ run through $28, 29, \ldots, 34$, each already less than $37$, so these are exactly the seven possible remainders.

Their sum is $28 + 29 + \cdots + 34 = 7 \cdot 31 = 217$.

**2.** Set $\mathcal{A}$ consists of $m$ consecutive integers whose sum is $2m$, and set $\mathcal{B}$ consists of $2m$ consecutive integers whose sum is $m$. The absolute value of the difference between the greatest element of $\mathcal{A}$ and the greatest element of $\mathcal{B}$ is $99$. Find $m$.

## Solution:

The $m$ integers of $\mathcal{A}$ have mean $\frac{2m}{m} = 2$, so they are centered at $2$; since the mean of consecutive integers is an integer only when there are an odd number of them, $m$ is odd and the greatest element of $\mathcal{A}$ is $2 + \frac{m-1}{2}$. The $2m$ integers of $\mathcal{B}$ have mean $\frac{1}{2}$, so they are $1 - m, \ldots, 0, 1, \ldots, m$, with greatest element $m$.

The condition is

$$\begin{aligned} \left|2 + \frac{m-1}{2} - m\right| &= \left|\frac{3-m}{2}\right| \\ &= 99, \end{aligned}$$

so $|3 - m| = 198$, giving $m = 201$ (since $m > 0$). Indeed $201$ is odd, as required, so $m = 201$.

**3.** A convex polyhedron $P$ has $26$ vertices, $60$ edges, and $36$ faces, $24$ of which are triangular, and $12$ of which are quadrilaterals. A space diagonal is a line segment connecting two non-adjacent vertices that do not belong to the same face. How many space diagonals does $P$ have?

## Solution:

Every pair of vertices determines exactly one of three things: an edge, a diagonal of a face, or a space diagonal. There are $\binom{26}{2} = 325$ pairs of vertices in all.

Of these, $60$ are edges. The $24$ triangular faces have no diagonals, while each of the $12$ quadrilateral faces has $2,$ for $24$ face diagonals (no two faces share a diagonal, since the polyhedron is convex).

The number of space diagonals is $325 - 60 - 24 = 241.$

**4.** A square has sides of length $2$. Set $\mathcal{S}$ is the set of all line segments that have length $2$ and whose endpoints are on adjacent sides of the square. The midpoints of the line segments in set $\mathcal{S}$ enclose a region whose area to the nearest hundredth is $k$. Find $100k$.

## Solution:

Let a segment $\overline{PQ}$ in $\mathcal{S}$ have endpoints on two sides meeting at corner $A$, and let $M$ be its midpoint. Triangle $PAQ$ is right-angled at $A$ with hypotenuse $PQ = 2$, and the median to the hypotenuse of a right triangle is half the hypotenuse, so $AM = 1$. Conversely every point at distance $1$ from a corner (between the two adjacent sides) is such a midpoint, so the midpoints form four quarter-circle arcs of radius $1$ centered at the corners of the square.

The region these arcs enclose is the square with the four quarter-disks removed, of area

$$4 - 4 \cdot \frac{\pi}{4} = 4 - \pi \approx 0.86.$$

Therefore $100k = 86$.

**5.** Alpha and Beta both took part in a two-day problem-solving competition. At the end of the second day, each had attempted questions worth a total of $500$ points. Alpha scored $160$ points out of $300$ points attempted on the first day, and scored $140$ points out of $200$ points attempted on the second day. Beta, who did not attempt $300$ points on the first day, had a positive integer score on each of the two days, and Beta's daily success ratio (points scored divided by points attempted) on each day was less than Alpha's on that day. Alpha's two-day success ratio was $\frac{300}{500} = \frac{3}{5}$. The largest possible two-day success ratio that Beta could have achieved is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m + n$?

## Solution:

Alpha's daily ratios were $\frac{160}{300} = \frac{8}{15}$ and $\frac{140}{200} = \frac{7}{10}$. Since $\frac{8}{15} < \frac{7}{10}$, Beta's score was less than $\frac{7}{10}$ of the points attempted on each day, so Beta's total score was less than $\frac{7}{10} \cdot 500 = 350$, hence at most $349$.

A total of $349$ is achievable: Beta can score $1$ out of $2$ points attempted on day one (and $\frac{1}{2} < \frac{8}{15}$) and $348$ out of $498$ on day two (and $\frac{348}{498} < \frac{7}{10}$ because $3480 < 3486$).

So Beta's largest possible two-day ratio is $\frac{349}{500}$, which is in lowest terms since $349$ is prime, and $m + n = 349 + 500 = 849.$

**6.** An integer is called *snakelike* if its decimal representation $a_1a_2a_3\dots a_k$ satisfies $a_i < a_{i+1}$ if $i$ is odd and $a_i > a_{i+1}$ if $i$ is even. How many snakelike integers between $1000$ and $9999$ have four distinct digits?

## Solution:

A four-digit snakelike number satisfies $a_1 < a_2 > a_3 < a_4$. First count the arrangements of any four distinct digits $w < x < y < z$ into this pattern. The largest digit $z$ must sit in position $2$ or $4$. If $z$ is in position $4$, the other three form $a_1 < a_2 > a_3$, so the largest of them takes position $2$ and the remaining two can go in either order: $2$ ways. If $z$ is in position $2$, any of the other three digits can be $a_1$, and then $a_3 < a_4$ fixes the rest: $3$ ways. So each set of four digits admits exactly $5$ snakelike orders.

If $0$ is not among the digits, all $5$ orders give valid numbers: $\binom{9}{4} \cdot 5 = 630$. If $0$ is among them, note $0$ must occupy position $1$ or $3$ (positions $2$ and $4$ must exceed a neighbor), and position $1$ is forbidden. With $0$ in position $3$, any of the other three digits can be $a_4$, and $a_1 < a_2$ fixes the rest, so $3$ of the $5$ orders survive: $\binom{9}{3} \cdot 3 = 252$.

The total is $630 + 252 = 882$.

**7.** Let $C$ be the coefficient of $x^2$ in the expansion of the product

$$(1-x)(1+2x)(1-3x)\cdots \\ \cdot (1+14x)(1-15x).$$

Find $|C|$.

## Solution:

Write the product as $\prod_{k=1}^{15}(1+a_k x)$ with $a_k = (-1)^k k$. An $x^2$ term arises by choosing the $x$-term from two factors, so $C = \sum_{i<j} a_i a_j$, and

$$C = \frac{\left(\sum a_k\right)^2 - \sum a_k^2}{2}.$$

The alternating sum is

$$\begin{aligned}&(-1+2)+(-3+4)\\ &\quad+\cdots+(-13+14)-15\\ &=7-15\\ &=-8,\end{aligned}$$

and

$$\begin{aligned}\sum a_k^2 &= 1^2+2^2+\cdots+15^2\\ &= \frac{15\cdot 16\cdot 31}{6}\\ &= 1240.\end{aligned}$$

Thus $C = \frac{64-1240}{2} = -588$, so $|C| = 588$.

**8.** Define a *regular $n$-pointed star* to be the union of $n$ line segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots,$ $\overline{P_nP_1}$ such that

• the points $P_1, P_2, \ldots, P_n$ are coplanar and no three of them are collinear,

• each of the $n$ line segments intersects at least one of the other line segments at a point other than an endpoint,

• all of the angles at $P_1, P_2, \ldots, P_n$ are congruent,

• all of the $n$ line segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_nP_1}$ are congruent, and

• the path $P_1P_2 \ldots P_nP_1$ turns counterclockwise at an angle of less than $180^\circ$ at each vertex.

There are no regular $3$-pointed, $4$-pointed, or $6$-pointed stars. All regular $5$-pointed stars are similar, but there are two non-similar regular $7$-pointed stars. How many non-similar regular $1000$-pointed stars are there?

## Solution:

The congruent angles and congruent segments force the vertices of a regular star to be equally spaced on a circle, visited by taking a constant step: number $n$ equally spaced points $0, 1, \ldots, n-1$ and connect every $d$th point. The path visits all $n$ points exactly when $\gcd(d, n) = 1$, and the segments actually cross (making a star rather than a convex polygon) exactly when $2 \leq d \leq n-2$. Steps $d$ and $n-d$ trace the same figure in opposite directions, while different values otherwise give non-similar stars, since a dilation matching the circles would have to match the turning angles.

For $n = 1000 = 2^3 \cdot 5^3$, the number of $d$ with $\gcd(d, 1000) = 1$ is $1000\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 400$. Removing $d = 1$ and $d = 999$ leaves $398$ values, which pair up as $\{d, 1000 - d\}$, so the number of non-similar regular $1000$-pointed stars is $\frac{398}{2} = 199.$

**9.** Let $ABC$ be a triangle with sides $3, 4,$ and $5,$ and $DEFG$ be a $6$-by-$7$ rectangle. A segment is drawn to divide triangle $ABC$ into a triangle $U_1$ and a trapezoid $V_1,$ and another segment is drawn to divide rectangle $DEFG$ into a triangle $U_2$ and a trapezoid $V_2$ such that $U_1$ is similar to $U_2$ and $V_1$ is similar to $V_2.$ The minimum value of the area of $U_1$ can be written in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n.$

## Solution:

A segment cuts the rectangle into a triangle and a trapezoid only if it runs from a vertex to a point on a nonadjacent side, so $U_2$ is a right triangle whose legs lie along two sides of the rectangle, one leg being a full side ($6$ or $7$). Since $U_1 \sim U_2,$ the cut in the $3$-$4$-$5$ right triangle $ABC$ must also produce a right triangle, so it is parallel to a leg, and then $U_1 \sim ABC.$ Hence $U_2$ is a $3$-$4$-$5$ triangle too: its legs are $6$ and $\frac{9}{2}$ (full side $6$) or $7$ and $\frac{21}{4}$ (full side $7$); the other orientations need legs $8$ or $\frac{28}{3}$, which do not fit.

In both cases the trapezoid $V_2$ has two right angles and an acute angle between the cut and its longer base with tangent $\frac{6}{\frac{9}{2}} = \frac{7}{\frac{21}{4}} = \frac{4}{3}.$ In triangle $ABC,$ a cut parallel to the leg of length $3$ gives $V_1$ an acute angle with tangent $\frac{4}{3}$, matching, while a cut parallel to the leg of length $4$ gives tangent $\frac{3}{4}$, which cannot match. So the cut is parallel to the side of length $3,$ and the parallel bases of $V_1$ are the cut segment $s$ and the side of length $3.$

Similarity of the trapezoids forces $s : 3$ to equal the ratio of the bases of $V_2,$ which is $\frac{7-\frac{9}{2}}{7} = \frac{5}{14}$ in the first case and $\frac{6-\frac{21}{4}}{6} = \frac{1}{8}$ in the second. Then $[U_1] = \left(\frac{s}{3}\right)^2 [ABC],$ giving $\left(\frac{5}{14}\right)^2 \cdot 6 = \frac{75}{98}$ or $\left(\frac{1}{8}\right)^2 \cdot 6 = \frac{3}{32}.$ The minimum is $\frac{3}{32}$, so $m+n = 3+32 = 35.$

**10.** A circle of radius $1$ is randomly placed in a $15$-by-$36$ rectangle $ABCD$ so that the circle lies completely within the rectangle. Given that the probability that the circle will not touch diagonal $\overline{AC}$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, find $m+n$.

## Solution:

Place $A=(0,0), B=(36,0), C=(36,15)$. For the circle to lie in the rectangle, its center must lie in the rectangle $[1,35]\times[1,14]$, of area $34\cdot 13=442$, and the center is uniformly distributed there. The diagonal $\overline{AC}$ lies on the line $5x-12y=0$, and the circle misses it exactly when the center's distance $\frac{|5x-12y|}{13}$ exceeds $1$, that is, $|5x-12y|>13$.

The line $5x-12y=13$ meets $y=1$ at $x=5$ and $x=35$ at $y=\frac{27}{2}$, so below the diagonal the favorable region is the right triangle with vertices $(5,1),(35,1),(35,\frac{27}{2})$, with legs $30$ and $\frac{25}{2}$ and area $\frac{1}{2}\cdot 30\cdot\frac{25}{2}=\frac{375}{2}$. Rotating $180^\circ$ about the rectangle's center $(18,\frac{15}{2})$, which lies on the diagonal, maps the inner rectangle and the diagonal to themselves, so the region above the diagonal has the same area.

The probability is $\frac{375}{442}$, and since $442=2\cdot 13\cdot 17$ shares no factor with $375=3\cdot 5^3$, we get $m+n=375+442=817$.

**11.** A solid in the shape of a right circular cone is $4$ inches tall and its base has a $3$-inch radius. The entire surface of the cone, including its base, is painted. A plane parallel to the base of the cone divides the cone into two solids, a smaller cone-shaped solid $\mathcal{C}$ and a frustum-shaped solid $\mathcal{F}$, in such a way that the ratio between the areas of the painted surfaces of $\mathcal{C}$ and $\mathcal{F}$ and the ratio between the volumes of $\mathcal{C}$ and $\mathcal{F}$ are both equal to $k$. Given that $k = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, find $m + n$.

## Solution:

The cone has radius $3$, height $4$, and slant height $5$, so its painted surface consists of lateral area $\pi \cdot 3 \cdot 5 = 15\pi$ and base area $9\pi$, totaling $24\pi$. Suppose the cut is at similarity ratio $t$, so $\mathcal{C}$ is a cone with radius $3t$ and slant height $5t$. Then $\mathcal{C}$'s painted surface is only its lateral area $15\pi t^2$, and $\mathcal{F}$'s painted surface is the rest, $24\pi - 15\pi t^2$. The volumes are in ratio $t^3$ to $1 - t^3$.

Setting the two ratios equal,

$$\frac{15t^2}{24 - 15t^2} = \frac{t^3}{1 - t^3},$$

so $15t^2(1 - t^3) = t^3(24 - 15t^2)$, which simplifies to $15t^2 = 24t^3$, giving $t = \frac{5}{8}$. Then $k = \frac{t^3}{1-t^3} = \frac{\frac{125}{512}}{\frac{387}{512}} = \frac{125}{387}$, which is in lowest terms since $387 = 3^2 \cdot 43$, so $m + n = 125 + 387 = 512$.

**12.** Let $\mathcal{S}$ be the set of ordered pairs $(x, y)$ such that $0 < x \leq 1, 0 < y \leq 1$, and $\left\lfloor \log_2 \left(\frac{1}{x}\right) \right\rfloor$ and $\left\lfloor \log_5 \left(\frac{1}{y}\right) \right\rfloor$ are both even. Given that the area of the graph of $\mathcal{S}$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, find $m + n$. The notation $\lfloor z \rfloor$ denotes the greatest integer that is less than or equal to $z$.

## Solution:

For $0 < x \leq 1$, the condition $\lfloor \log_2(\frac{1}{x}) \rfloor = 2k$ (for an integer $k \geq 0$) means $2k \leq \log_2(\frac{1}{x}) < 2k + 1$, i.e. $x \in \left(2^{-2k-1}, 2^{-2k}\right]$. These intervals have total length

$$\sum_{k\geq 0} 2^{-2k-1} = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}.$$

Similarly, $\lfloor \log_5(\frac{1}{y}) \rfloor$ is even for $y \in \left(5^{-2k-1}, 5^{-2k}\right]$, intervals of total length $\sum_{k\geq 0} \frac{4}{5} \cdot 25^{-k} = \frac{\frac{4}{5}}{1 - \frac{1}{25}} = \frac{5}{6}$.

The graph of $\mathcal{S}$ is the product of these two sets, so its area is $\frac{2}{3} \cdot \frac{5}{6} = \frac{5}{9}$, and $m + n = 5 + 9 = 14$.

**13.** The polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$$

has $34$ complex zeros of the form $z_k = r_k[\cos(2\pi\alpha_k) + i\sin(2\pi\alpha_k)]$, $k = 1, 2, 3, \ldots, 34$, with $0 < \alpha_1 \le \alpha_2 \le \alpha_3 \le \cdots \le \alpha_{34} < 1$ and $r_k > 0$. Given that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, find $m + n$.

## Solution:

For $x \neq 1$, write $1 + x + \cdots + x^{17} = \frac{x^{18}-1}{x-1}$, so

$$\begin{aligned}(x-1)^2 P(x) &= (x^{18}-1)^2 - x^{17}(x-1)^2 \\ &= x^{36} - x^{19} - x^{17} + 1 \\ &= (x^{19}-1)(x^{17}-1).\end{aligned}$$

Hence the zeros of $P$ are the $34$ complex numbers other than $1$ satisfying $x^{17} = 1$ or $x^{19} = 1$; all lie on the unit circle, with angles $\alpha = \frac{k}{17}$ $(k = 1, \ldots, 16)$ and $\alpha = \frac{k}{19}$ $(k = 1, \ldots, 18)$.

The five smallest of these angles are

$$\frac{1}{19} < \frac{1}{17} < \frac{2}{19} < \frac{2}{17} < \frac{3}{19},$$

whose sum is $\frac{6}{19} + \frac{3}{17} = \frac{102+57}{323} = \frac{159}{323}$. Since $159 = 3 \cdot 53$ and $323 = 17 \cdot 19$, this is in lowest terms, and $m + n = 159 + 323 = 482$.

**14.** A unicorn is tethered by a $20$-foot silver rope to the base of a magician's cylindrical tower whose radius is $8$ feet. The rope is attached to the tower at ground level and to the unicorn at a height of $4$ feet. The unicorn has pulled the rope taut, the end of the rope is $4$ feet from the nearest point on the tower, and the length of the rope that is touching the tower is $\frac{a-\sqrt{b}}{c}$ feet, where $a, b,$ and $c$ are positive integers, and $c$ is prime. Find $a + b + c.$

## Solution:

The rope runs from its anchor $A$ at the base of the tower, hugs the wall up to a point $P$, then goes straight to its end $Q$, which is at height $4$ and at distance $8 + 4 = 12$ from the tower's axis. Unroll the cylinder's wall into a plane: a taut rope becomes a single straight segment of length $20$ rising $4$ feet, so its horizontal projection has length $\sqrt{20^2 - 4^2} = 8\sqrt{6},$ and every piece of the rope has the same ratio $\frac{20}{8\sqrt{6}} = \frac{5}{2\sqrt{6}}$ of length to horizontal projection.

Viewed from above, the free portion $PQ$ is tangent to the circle of radius $8$ at $P$ from a point at distance $12,$ so its horizontal projection has length $\sqrt{12^2 - 8^2} = 4\sqrt{5}.$ Therefore

$$\begin{aligned} PQ &= \frac{5}{2\sqrt{6}} \cdot 4\sqrt{5} \\ &= \frac{10\sqrt{5}}{\sqrt{6}} \\ &= \frac{5\sqrt{30}}{3}. \end{aligned}$$

The rope touching the tower has length $20 - \frac{5\sqrt{30}}{3} = \frac{60-\sqrt{750}}{3}$, and $c = 3$ is prime, so $a + b + c = 60 + 750 + 3 = 813.$

**15.** For all positive integers $x$, let

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ \frac{x}{10} & \text{if } x \text{ is divisible by } 10, \\ x+1 & \text{otherwise,} \end{cases}$$

and define a sequence as follows: $x_1 = x$ and $x_{n+1} = f(x_n)$ for all positive integers $n$. Let $d(x)$ be the smallest $n$ such that $x_n = 1$. (For example, $d(100) = 3$ and $d(87) = 7$.) Let $m$ be the number of positive integers $x$ such that $d(x) = 20$. Find the sum of the distinct prime factors of $m$.

## Solution:

Work backwards: $f(z') = z$ for $z' = 10z$ (always) and for $z' = z - 1$ (provided $z - 1$ is not a multiple of $10$ and $z - 1 \neq 1$, i.e. $z$ does not end in $1$ and $z \neq 2$). So the integers with $d(x) = n$ form column $n$ of a tree rooted at $1$ : the columns begin $\{1\}$, $\{10\}, \{9, 100\}, \{8, 90, 99, 1000\}$, and every vertex has two children except $2$ and the vertices ending in $1$, which have only the child $10z$.

Locate those one-child vertices. Since $2 \to 3 \to \cdots \to 10 \to 1$, the vertex $2$ sits in column $10$. A vertex ending in $1$ is reached by subtracting $1$ nine times from a vertex ending in $0$, so such vertices sit $9$ columns after the multiples of $10$ in the tree. Columns $2$ through $10$ double perfectly (no one-child vertices occur that early), so column $j$ has $2^{j-2}$ vertices, of which the multiples of $10$ — the children $10z$ of column $j - 1$ — number $2^{j-3}$. Hence for $12 \leq k \leq 19$, column $k$ contains $2^{k-12}$ vertices ending in $1$ (column $11$ has none, because the multiple of $10$ in column $2$ is $10$ itself, whose descendant $1$ is excluded — that exclusion is exactly the missing child of $2$).

A one-child vertex in column $k$ removes $2^{19-k}$ of the potential $2^{18}$ vertices from column $20$. Therefore

$$
\begin{aligned}
m &= 2^{18} - 2^9 \\
&\quad - \sum_{k=12}^{19} 2^{k-12} \cdot 2^{19-k} \\
&= 2^{18} - 2^9 - 8 \cdot 2^7 \\
&= 2^9(2^9 - 1 - 2) \\
&= 2^9 \cdot 509.
\end{aligned}
$$

Since $509$ is prime, the sum of the distinct prime factors of $m$ is $2 + 509 = 511.$

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