

2005 AIME II Solutions

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1. A game uses a deck of n different cards, where n is an integer and $n \geq 6$. The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. Find n .



Solution:

The condition says $\binom{n}{6} = 6\binom{n}{3}$. Dividing the binomial coefficients,

$$\frac{\binom{n}{6}}{\binom{n}{3}} = \frac{(n-3)(n-4)(n-5)}{6 \cdot 5 \cdot 4} = 6,$$

$$\text{so } (n-3)(n-4)(n-5) = 720 = 10 \cdot 9 \cdot 8.$$

Since the product $(n-3)(n-4)(n-5)$ is increasing in n , the only solution is $n-3 = 10$, that is, $n = 13$.

2. A hotel packed a breakfast for each of three guests. Each breakfast should have consisted of three types of rolls, one each of nut, cheese, and fruit rolls. The preparer wrapped each of the nine rolls, and, once they were wrapped, the rolls were indistinguishable from one another. She then randomly put three rolls in a bag for each of the guests. Given that the probability that each guest got one roll of each type is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.



Solution:

Fill the first guest's bag one roll at a time. The first roll can be anything; the second must avoid the 2 remaining rolls of the first roll's type, succeeding with probability $\frac{6}{8}$; and the third must be one of the 3 rolls of the missing type among the remaining 7. So the first bag has one roll of each type with probability $\frac{6}{8} \cdot \frac{3}{7} = \frac{9}{28}$.

Given that, six rolls remain, two of each type, and the same argument gives $\frac{4}{5} \cdot \frac{2}{4} = \frac{2}{5}$ for the second bag. The third bag is then automatically one of each type. The probability is $\frac{9}{28} \cdot \frac{2}{5} = \frac{9}{70}$, so $m + n = 9 + 70 = 79$.

3. An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has sum 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution:

Let the original series have first term a and ratio r , so $\frac{a}{1-r} = 2005$. The squared series is geometric with first term a^2 and ratio r^2 , so

$$\begin{aligned}\frac{a^2}{1-r^2} &= \frac{a}{1-r} \cdot \frac{a}{1+r} \\ &= 2005 \cdot \frac{a}{1+r} \\ &= 10 \cdot 2005,\end{aligned}$$

which gives $\frac{a}{1+r} = 10$.

Dividing the two equations, $\frac{1+r}{1-r} = \frac{2005}{10}$, so $2(1+r) = 401(1-r)$, giving $403r = 399$ and $r = \frac{399}{403}$. Since $399 = 3 \cdot 7 \cdot 19$ and $403 = 13 \cdot 31$, the fraction is in lowest terms, and $m + n = 399 + 403 = 802$.

4. Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$.



Solution:

From the factorizations $10^{10} = 2^{10}5^{10}$, $15^7 = 3^75^7$, and $18^{11} = 2^{11}3^{22}$, the divisor counts are $11 \cdot 11 = 121$, $8 \cdot 8 = 64$, and $12 \cdot 23 = 276$.

The divisors common to two of the numbers are exactly the divisors of their gcd: $\gcd(10^{10}, 15^7) = 5^7$ has 8 divisors, $\gcd(10^{10}, 18^{11}) = 2^{10}$ has 11, and $\gcd(15^7, 18^{11}) = 3^7$ has 8. Only 1 divides all three numbers.

By inclusion-exclusion, the count is $121 + 64 + 276 - 8 - 11 - 8 + 1 = 435$.

5. Determine the number of ordered pairs (a, b) of integers such that $\log_a b + 6 \log_b a = 5$, $2 \leq a \leq 2005$, and $2 \leq b \leq 2005$.



Solution:

Let $x = \log_a b$. Since $\log_b a = \frac{1}{x}$, the equation becomes $x + \frac{6}{x} = 5$, i.e. $x^2 - 5x + 6 = 0$, so $x = 2$ or $x = 3$. That means $b = a^2$ or $b = a^3$.

For $b = a^2 \leq 2005$ we need $2 \leq a \leq 44$ (since $44^2 = 1936$ and $45^2 = 2025$), giving 43 pairs. For $b = a^3 \leq 2005$ we need $2 \leq a \leq 12$ (since $12^3 = 1728$ and $13^3 = 2197$), giving 11 pairs.

In total there are $43 + 11 = 54$ ordered pairs.

6. The cards in a stack of $2n$ cards are numbered consecutively from 1 through $2n$ from top to bottom. The top n cards are removed, kept in order, and form pile A . The remaining cards form pile B . The cards are now restacked into a single stack by taking cards alternately from the tops of pile B and pile A , respectively. In this process, card number $(n + 1)$ is the bottom card of the new stack, card number 1 is on top of this card, and so on, until piles A and B are exhausted. If, after the restacking process, at least one card from each pile occupies the same position that it occupied in the original stack, the stack is called *magical*. For example, eight cards form a magical stack because cards number 3 and number 6 retain their original positions. Find the number of cards in the magical stack in which card number 131 retains its original position.



Solution:

The new stack, read from the bottom up, is $n + 1, 1, n + 2, 2, \dots, 2n, n$. So pile B 's cards occupy the even positions from the top in reverse order, and pile A 's cards occupy the odd positions in reverse order: a card at original position $i \leq n$ (pile A) moves to position $2(n - i) + 1$, while a card at position $i > n$ (pile B) moves to position $2(2n - i) + 2$.

Since 131 is odd, card 131 can keep its position only if it comes from pile A , so $131 = 2(n - 131) + 1$, which gives $n = 196$. Indeed $131 \leq 196$, and the stack is magical because card 262 from pile B also stays fixed: $2(2n - 262) + 2 = 262$. The stack has $2n = 392$ cards.

7. Let

$$x = \frac{4}{(\sqrt{5} + 1)(\sqrt[4]{5} + 1)(\sqrt[8]{5} + 1)(\sqrt[16]{5} + 1)}.$$

Find $(x + 1)^{48}$.



Solution:

Let $y = \sqrt[16]{5}$. Multiplying the numerator and denominator by $y - 1$ telescopes the denominator by repeated difference of squares:

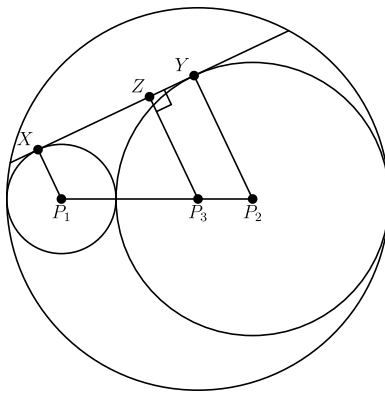
$$\begin{aligned} x &= \frac{4(y - 1)}{(y^8 + 1)(y^4 + 1)(y^2 + 1)(y + 1)(y - 1)} \\ &= \frac{4(y - 1)}{y^{16} - 1} \\ &= \frac{4(y - 1)}{4} \\ &= y - 1. \end{aligned}$$

Hence $x + 1 = y = 5^{\frac{1}{16}}$, and $(x + 1)^{48} = 5^{\frac{48}{16}} = 5^3 = 125$.

8. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are externally tangent, and they are both internally tangent to circle $\mathcal{C}_3$. The radii of $\mathcal{C}_1$ and $\mathcal{C}_2$ are 4 and 10, respectively, and the centers of the three circles are all collinear. A chord of $\mathcal{C}_3$ is also a common external tangent of $\mathcal{C}_1$ and $\mathcal{C}_2$. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$, where m , n , and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.



Solution:



Let P_1, P_2, P_3 be the centers of the circles and R the radius of $\mathcal{C}_3$. External tangency gives $P_1P_2 = 4 + 10 = 14$, and internal tangency gives $P_3P_1 = R - 4$ and $P_3P_2 = R - 10$. Since the centers are collinear, $(R - 4) + (R - 10) = 14$, so $R = 14$ and P_3 lies on $\overline{P_1P_2}$ with $P_3P_1 = 10$ and $P_3P_2 = 4$.

Drop perpendiculars P_1X, P_2Y, P_3Z to the chord, so $P_1X = 4, P_2Y = 10$, and Z is the midpoint of the chord. The distance from a point moving along line P_1P_2 to the tangent line changes linearly, and P_3 is $\frac{10}{14}$ of the way from P_1 to P_2 , so

$$P_3Z = 4 + \frac{10}{14}(10 - 4) = \frac{58}{7}.$$

The half-chord is $\sqrt{14^2 - \left(\frac{58}{7}\right)^2} = \frac{\sqrt{9604 - 3364}}{7} = \frac{4\sqrt{390}}{7}$, so the chord has length $\frac{8\sqrt{390}}{7}$. Since $390 = 2 \cdot 3 \cdot 5 \cdot 13$ is squarefree and $\gcd(8, 7) = 1$, the answer is $m + n + p = 8 + 390 + 7 = 405$.

9. For how many positive integers n less than or equal to 1000 is

$$\begin{aligned}(\sin t + i \cos t)^n \\ = \sin nt + i \cos nt\end{aligned}$$

true for all real t ?



Solution:

Since $\sin t + i \cos t = i(\cos t - i \sin t)$ and $\sin nt + i \cos nt = i(\cos nt - i \sin nt)$, de Moivre's theorem (applied to angle $-t$) gives

$$\begin{aligned}(\sin t + i \cos t)^n \\ = i^n (\cos t - i \sin t)^n \\ = i^n (\cos nt - i \sin nt).\end{aligned}$$

So the equation holds for all real t exactly when $i^n = i$, that is, when $n \equiv 1 \pmod{4}$. The values $n = 1, 5, 9, \dots, 997$ give exactly 250 positive integers up to 1000.

10. Given that $\mathcal{O}$ is a regular octahedron, that $\mathcal{C}$ is the cube whose vertices are the centers of the faces of $\mathcal{O}$, and that the ratio of the volume of $\mathcal{O}$ to that of $\mathcal{C}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.



Solution:

Place the octahedron's vertices at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. It is two square pyramids glued along the square with vertices $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$, which has area 2, and each pyramid has height 1, so

$$V_{\mathcal{O}} = 2 \cdot \frac{1}{3} \cdot 2 \cdot 1 = \frac{4}{3}.$$

Each face centroid is the average of that face's three vertices, e.g. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, so the cube has vertices $(\pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3})$. Its edge is $\frac{2}{3}$ and its volume is $\frac{8}{27}$.

The ratio is $\frac{\frac{4}{3}}{\frac{8}{27}} = \frac{9}{2}$, so $m + n = 9 + 2 = 11$.

11. Let m be a positive integer, and let $a_0, a_1, \dots, a_m$ be a sequence of real numbers such that $a_0 = 37, a_1 = 72, a_m = 0$, and

$$a_{k+1} = a_{k-1} - \frac{3}{a_k}$$

for $k = 1, 2, \dots, m - 1$. Find m .



Solution:

Multiplying the recurrence by a_k gives $a_{k+1}a_k = a_k a_{k-1} - 3$, so the products $b_k = a_k a_{k-1}$ form an arithmetic sequence with common difference -3 . Since $b_1 = 72 \cdot 37 = 2664 = 3 \cdot 888$, we get

$$\begin{aligned} b_k &= 2664 - 3(k - 1) \\ &= 3(889 - k). \end{aligned}$$

Thus $b_k > 0$ for $k \leq 888$, so no term before a_{889} can vanish (and the recurrence never divides by zero), while $b_{889} = a_{889}a_{888} = 0$ with $a_{888} \neq 0$. Hence $a_{889} = 0$, and $m = 889$.

12. Square $ABCD$ has center O , $AB = 900$, E and F are on $\overline{AB}$ with $AE < BF$ and E between A and F , $m\angle EOF = 45^\circ$, and $EF = 400$. Given that $BF = p + q\sqrt{r}$, where p, q , and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.



Solution:

Let G be the midpoint of $\overline{AB}$, so $OG \perp AB$ and $OG = 450$. With $\alpha = \angle EOG$ and $\beta = \angle FOG$ on either side of ray OG , we have $EG = 450 \tan \alpha$, $FG = 450 \tan \beta$, and $\alpha + \beta = 45^\circ$. From $EG + FG = EF = 400$, we get $\tan \alpha + \tan \beta = \frac{8}{9}$.

The tangent addition formula gives

$$1 = \tan 45^\circ = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta},$$

so $\tan \alpha \tan \beta = 1 - \frac{8}{9} = \frac{1}{9}$. Hence $\tan \alpha$ and $\tan \beta$ are the roots of $9t^2 - 8t + 1 = 0$, namely $\frac{4 \pm \sqrt{7}}{9}$.

Since $AE = 450 - EG$ and $BF = 450 - FG$, the condition $AE < BF$ means $EG > FG$, so $\tan \beta = \frac{4 - \sqrt{7}}{9}$. Then $BF = 450 - 450 \cdot \frac{4 - \sqrt{7}}{9} = 250 + 50\sqrt{7}$, and $p + q + r = 250 + 50 + 7 = 307$.

13. Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that the equation $P(n) = n + 3$ has two distinct integer solutions n_1 and n_2 , find the product $n_1 \cdot n_2$.



Solution:

Let $S(x) = P(x) - x - 3$, so $S(17) = S(24) = -10$. Since $S(x) + 10$ has integer coefficients and vanishes at 17 and 24,

$$S(x) = -10 + (x - 17)(x - 24)Q(x)$$

for some polynomial Q with integer coefficients.

If $P(n) = n + 3$ for an integer n , then $(n - 17)(n - 24)Q(n) = 10$, so $(n - 17)(n - 24)$ divides 10. The factors $n - 17$ and $n - 24$ are integers differing by 7 whose product divides 10, so they are $\{2, -5\}$ or $\{5, -2\}$, giving $n = 19$ and $n = 22$. Both occur, for example, for $P(x) = x - 7 - (x - 17)(x - 24)$.

Hence $n_1 \cdot n_2 = 19 \cdot 22 = 418$.

14. In triangle ABC , $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$, where p and q are relatively prime positive integers, find q .



Solution:

A cevian AD splits the opposite side in the ratio

$$\begin{aligned} \frac{BD}{DC} &= \frac{[ABD]}{[ACD]} \\ &= \frac{\frac{1}{2}AB \cdot AD \sin \angle BAD}{\frac{1}{2}AC \cdot AD \sin \angle CAD} \\ &= \frac{AB \sin \angle BAD}{AC \sin \angle CAD}, \end{aligned}$$

and similarly $\frac{BE}{EC} = \frac{AB \sin \angle BAE}{AC \sin \angle CAE}$.

Since $\angle BAE = \angle CAD$, we also have $\angle BAD = \angle CAE$ (each is that common angle plus $\angle EAD$), so multiplying the two ratios cancels all the sines: $\frac{BD}{DC} \cdot \frac{BE}{EC} = \frac{AB^2}{AC^2}$. With $BD = 9$, $DC = 6$, $AB = 13$, and $AC = 14$, this gives

$$\frac{BE}{EC} = \frac{13^2}{14^2} \cdot \frac{6}{9} = \frac{169}{294}.$$

Hence $BE = 15 \cdot \frac{169}{169+294} = \frac{2535}{463}$. Since 463 is prime and does not divide 2535 = $3 \cdot 5 \cdot 13^2$, the fraction is in lowest terms, and $q = 463$.

15. Let ω_1 and ω_2 denote the circles $x^2 + y^2 + 10x - 24y - 87 = 0$ and $x^2 + y^2 - 10x - 24y + 153 = 0$, respectively. Let m be the smallest positive value of a for which the line $y = ax$ contains the center of a circle that is internally tangent to ω_1 and externally tangent to ω_2 . Given that $m^2 = \frac{p}{q}$, where p and q are relatively prime positive integers, find $p + q$.



Solution:

Completing the square gives $\omega_1: (x + 5)^2 + (y - 12)^2 = 256$ and $\omega_2: (x - 5)^2 + (y - 12)^2 = 16$, with centers $F_1 = (-5, 12)$ and $F_2 = (5, 12)$ and radii 16 and 4. If a circle with center P and radius r is internally tangent to ω_1 and externally tangent to ω_2 , then $PF_1 = 16 - r$ and $PF_2 = 4 + r$, so $PF_1 + PF_2 = 20$.

Thus P lies on the ellipse with foci F_1 and F_2 and major axis 20 : the semimajor axis is 10, the center-to-focus distance is 5, so the semiminor axis squared is $100 - 25 = 75$, giving

$$\frac{x^2}{100} + \frac{(y - 12)^2}{75} = 1,$$

i.e. $3x^2 + 4y^2 - 96y + 576 = 300$. Substituting $y = ax$ yields $(3 + 4a^2)x^2 - 96ax + 276 = 0$.

The line $y = ax$ contains such a center exactly when this quadratic has a real root, i.e. $(96a)^2 - 4 \cdot 276(3 + 4a^2) \geq 0$, which simplifies to $4800a^2 \geq 3312$, so $a^2 \geq \frac{69}{100}$.

The smallest positive such a has $m^2 = \frac{69}{100}$, and $p + q = 69 + 100 = 169$.

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