# 2006 AIME I Solutions

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**1.** In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18$, $BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

## Solution:

Triangle $ABC$ is right-angled at $B$, so $AC^2 = 18^2 + 21^2 = 765$. Triangle $ACD$ is right-angled at $C$, so $DA^2 = AC^2 + CD^2 = 765 + 196 = 961$, giving $DA = 31$.

The perimeter is $18 + 21 + 14 + 31 = 84$.

**2.** Let set $\mathcal{A}$ be a $90$-element subset of $\{1, 2, 3, \ldots, 100\}$, and let $S$ be the sum of the elements of $\mathcal{A}$. Find the number of possible values of $S$.

## Solution:

The smallest possible sum is $1 + 2 + \cdots + 90 = 4095$, and the largest is $11 + 12 + \cdots + 100 = 4995$.

Every integer in between also occurs. Suppose $\mathcal{A}$ has sum $S \leq 4994$, and let $k$ be the smallest element of $\mathcal{A}$ with $k + 1 \notin \mathcal{A}$. If $k$ were $100$, then $\mathcal{A}$ would be a block of consecutive integers ending at $100$, namely $\{11, \ldots, 100\}$, whose sum exceeds $4994$. So $k \neq 100$, and replacing $k$ by $k + 1$ produces a $90$-element subset with sum $S + 1$.

Hence $S$ takes every value from $4095$ to $4995$, for $4995 - 4095 + 1 = 901$ possible values.

**3.** Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

## Solution:

Let $d$ be the leftmost digit and $n$ the integer that remains after deleting it, so the original integer is $d \cdot 10^p + n$ for some positive integer $p$. The condition says $d \cdot 10^p + n = 29n$, so $d \cdot 10^p = 28n$.

Since $7 \mid 28n$ but $7 \nmid 10^p$, the digit $d$ must be a multiple of $7$, so $d = 7$. Then $10^p = 4n$, giving $n = 25 \cdot 10^{p-2}$, which requires $p \geq 2$. The smallest case is $p = 2, n = 25$.

The least such integer is $725$, and indeed $725 = 29 \cdot 25$.

**4.** Let $N$ be the number of consecutive $0$'s at the right end of the decimal representation of the product $1!\,2!\,3!\,4!\cdots 99!\,100!$. Find the remainder when $N$ is divided by $1000$.

## Solution:

Factors of $2$ are plentiful, so $N$ is the exponent of $5$ in the product. Each integer $j$ with $1 \leq j \leq 100$ appears as a factor in exactly $101 - j$ of the factorials, namely $j!, (j+1)!, \ldots, 100!$.

Every multiple of $5$ contributes one factor of $5$ per appearance, and every multiple of $25$ contributes one more. Over $j = 5, 10, \ldots, 100$ the appearances total $96 + 91 + \cdots + 1 = \frac{20 \cdot 97}{2} = 970$, and over $j = 25, 50, 75, 100$ they total $76 + 51 + 26 + 1 = 154$.

Hence $N = 970 + 154 = 1124$, and the remainder upon division by $1000$ is $124$.

**5.** The number

$$\sqrt{104\sqrt{6} + 468\sqrt{10} + 144\sqrt{15} + 2006}$$

can be written as $a\sqrt{2} + b\sqrt{3} + c\sqrt{5}$, where $a, b,$ and $c$ are positive integers. Find $a \cdot b \cdot c$.

## Solution:

Squaring $a\sqrt{2} + b\sqrt{3} + c\sqrt{5}$ gives

$$\begin{aligned} &2a^2 + 3b^2 + 5c^2 \\ &\quad + 2ab\sqrt{6} + 2ac\sqrt{10} \\ &\quad + 2bc\sqrt{15}. \end{aligned}$$

Matching coefficients yields $2ab = 104, 2ac = 468, 2bc = 144,$ that is $ab = 52,$ $ac = 234, bc = 72,$ along with $2a^2 + 3b^2 + 5c^2 = 2006.$

Then

$$\begin{aligned} (abc)^2 &= (ab)(ac)(bc) \\ &= 52 \cdot 234 \cdot 72 \\ &= 876096 = 936^2, \end{aligned}$$

so $abc = 936.$ As a check, $a = \frac{abc}{bc} = 13, b = \frac{abc}{ac} = 4, c = \frac{abc}{ab} = 18,$ and

$$\begin{aligned} &2 \cdot 169 + 3 \cdot 16 + 5 \cdot 324 \\ &= 338 + 48 + 1620 \\ &= 2006, \end{aligned}$$

as required.

Therefore $a \cdot b \cdot c = 936.$

**6.** Let $\mathcal{S}$ be the set of real numbers that can be represented as repeating decimals of the form $0.\overline{abc}$ where $a, b, c$ are distinct digits. Find the sum of the elements of $\mathcal{S}$.

## Solution:

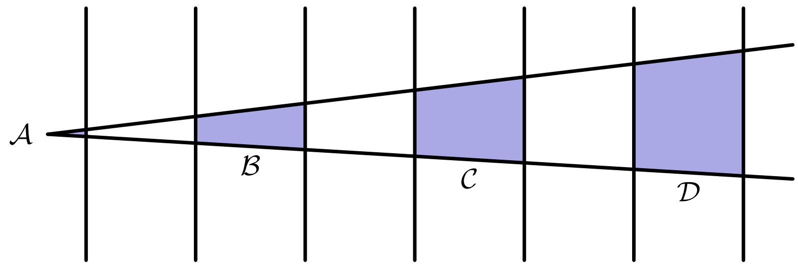
Each element equals $0.\overline{abc} = \frac{100a+10b+c}{999}$, and there are $10 \cdot 9 \cdot 8 = 720$ ordered triples of distinct digits. By symmetry, each digit $0$ through $9$ appears in each of the three positions exactly $\frac{720}{10} = 72$ times.

The numerators therefore total

$$\begin{aligned} &72(0+1+\cdots+9)\\ &\quad\cdot(100+10+1)\\ &= 72 \cdot 45 \cdot 111 = 359640, \end{aligned}$$

so the sum of the elements is $\frac{359640}{999} = 360.$

**7.** An angle is drawn on a set of equally spaced parallel lines as shown. The ratio of the area of shaded region $\mathcal{C}$ to the area of shaded region $\mathcal{B}$ is $\frac{11}{5}$. Find the ratio of the area of shaded region $\mathcal{D}$ to the area of shaded region $\mathcal{A}$.



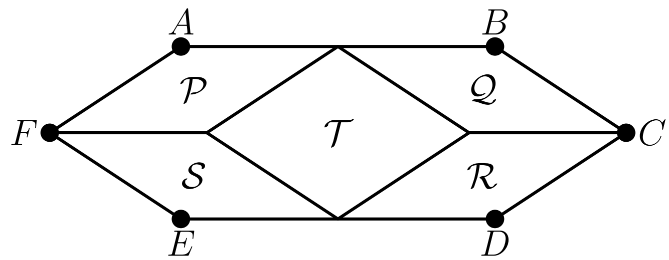

## Solution:

Take the spacing between consecutive lines as the unit, and let $x$ be the distance from the vertex of the angle to the first line. The triangle cut off by the $j$th line is similar to triangle $\mathcal{A}$ with ratio $\frac{x+j-1}{x}$, so its area is proportional to $(x+j-1)^2$, and the strip between lines $j$ and $j+1$ has area proportional to $(x+j)^2 - (x+j-1)^2 = 2x + 2j - 1$.

Regions $\mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ are the strips beginning at lines $2, 4$, and $6$, with areas proportional to $2x+3, 2x+7$, and $2x+11$. The given ratio yields $\frac{2x+7}{2x+3} = \frac{11}{5}$, so $10x + 35 = 22x + 33$ and $x = \frac{1}{6}$.

Since $\mathcal{A}$ has area proportional to $x^2$, the requested ratio is $\frac{2x+11}{x^2} = \frac{\frac{34}{3}}{\frac{1}{36}} = 408.$

**8.** Hexagon $ABCDEF$ is divided into five rhombuses, $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$, and $\mathcal{T}$, as shown. Rhombuses $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, and $\mathcal{S}$ are congruent, and each has area $\sqrt{2006}$. Let $K$ be the area of rhombus $\mathcal{T}$. Given that $K$ is a positive integer, find the number of possible values for $K$.




## Solution:

Since $\mathcal{T}$ shares a side with each of the other rhombuses, all five have the same side length $z$. Let $Y$ be the vertex of $\mathcal{T}$ on $\overline{AB}$, and let $\alpha$ be the angle of $\mathcal{P}$ at $Y$. Then each congruent rhombus has area $z^2 \sin\alpha = \sqrt{2006}$. The angles of $\mathcal{P}, \mathcal{T}$, and $\mathcal{Q}$ at $Y$ lie along the line $AB$, and by symmetry $\mathcal{Q}$'s angle there also equals $\alpha$, so $\mathcal{T}$'s angle is $180^\circ - 2\alpha$. Hence

$$\begin{aligned} K &= z^2 \sin(180^\circ - 2\alpha) \\ &= z^2 \sin 2\alpha \\ &= 2z^2 \sin\alpha \cos\alpha \\ &= 2\sqrt{2006}\,\cos\alpha. \end{aligned}$$

As $\alpha$ ranges over $(0^\circ, 90^\circ)$, the value $\cos\alpha$ takes every value in $(0,1)$, so $K$ takes every value in $\left(0, \sqrt{8024}\right)$. Since $89^2 = 7921 < 8024 < 8100 = 90^2$, the possible positive integer values are $1, 2, \ldots, 89$ : there are $89$ of them.

**9.** The sequence $a_1, a_2, ...$ is geometric with $a_1 = a$ and common ratio $r$, where $a$ and $r$ are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs $(a, r)$.

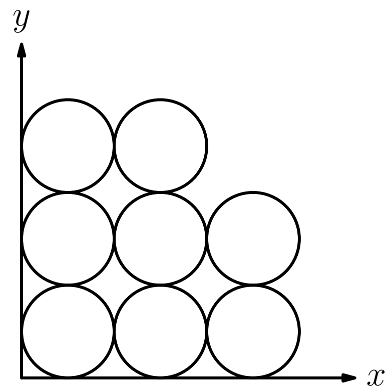
## Solution:

The sum of the logarithms is $\log_8(a_1 a_2 \cdots a_{12}) = \log_8\left(a^{12} r^{66}\right)$, so $a^{12} r^{66} = 8^{2006} = 2^{6018}$, which gives $a^2 r^{11} = 2^{1003}$.

Thus $a$ and $r$ are powers of $2$ : write $a = 2^x$ and $r = 2^y$ with integers $x, y \geq 0$ and $2x + 11y = 1003$. Since $2x$ is even and $1003$ is odd, $y$ must be odd, say $y = 2k - 1$ for $k \geq 1$. Then $x = 507 - 11k \geq 0$ exactly when $k \leq \lfloor \frac{507}{11} \rfloor = 46$.

Each $k = 1, 2, \ldots, 46$ gives one pair, so there are $46$ ordered pairs $(a, r)$.

**10.** Eight circles of diameter $1$ are packed in the first quadrant of the coordinate plane as shown. Let region $\mathcal{R}$ be the union of the eight circular regions. Line $\ell$, with slope $3$, divides $\mathcal{R}$ into two regions of equal area. Line $\ell$'s equation can be expressed in the form $ax = by + c$, where $a, b,$ and $c$ are positive integers whose greatest common divisor is $1$. Find $a^2 + b^2 + c^2$.




## Solution:

The circles have radius $\frac{1}{2}$ and centers at $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}\right)$, $\left(\frac{5}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(\frac{3}{2}, \frac{3}{2}\right)$, $\left(\frac{5}{2}, \frac{3}{2}\right)$, $\left(\frac{1}{2}, \frac{5}{2}\right)$, and $\left(\frac{3}{2}, \frac{5}{2}\right)$. The pair of circles tangent at $A = \left(1, \frac{1}{2}\right)$ is symmetric about $A$, so any line through $A$ bisects that pair's area; similarly for the pair tangent at $B = \left(\frac{3}{2}, 2\right)$. The line $AB$ has slope $\frac{2-\frac{1}{2}}{\frac{3}{2}-1} = 3.$

Line $AB$ misses the remaining four circles entirely, and exactly two of them lie on each side of it, so it divides $\mathcal{R}$ into two regions of equal area. Sliding a slope-$3$ line strictly shifts area from one side to the other, so $\ell$ must be this line.

Its equation is $y - \frac{1}{2} = 3(x - 1)$, that is, $6x = 2y + 5$. With $\gcd(6, 2, 5) = 1$, the answer is $a^2 + b^2 + c^2 = 36 + 4 + 25 = 65.$

**11.** A collection of $8$ cubes consists of one cube with edge-length $k$ for each integer $k$, $1 \le k \le 8$. A tower is to be built using all $8$ cubes according to the rules:

• Any cube may be the bottom cube in the tower.

• The cube immediately on top of a cube with edge-length $k$ must have edge-length at most $k+2$.

Let $T$ be the number of different towers that can be constructed. What is the remainder when $T$ is divided by $1000$?

## Solution:

Let $S(n)$ be the number of legal towers using the cubes of edge-lengths $1, \ldots, n$. Given a legal tower of $n \ge 2$ cubes, cube $n+1$ can be inserted in exactly three places: at the bottom, immediately on top of cube $n$, or immediately on top of cube $n-1$ (anywhere else it would rest on a cube of edge-length at most $n-2$, violating the rule). Each insertion stays legal, because the cube that ends up on top of cube $n+1$ has edge-length at most $n < (n+1)+2$. Conversely, deleting cube $n+1$ from a legal tower of $n+1$ cubes leaves a legal tower: the cube that was above it, of edge-length at most $n$, lands on a cube of edge-length at least $n-1$.

Hence $S(n+1) = 3S(n)$ for $n \ge 2$. Since $S(2) = 2$ (either of the two cubes may be on top), we get $T = S(8) = 2 \cdot 3^6 = 1458$, and the remainder is $458$.

**12.** Find the sum of the values of $x$ such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where $x$ is measured in degrees and $100 < x < 200$.

## Solution:

By the product-to-sum identity, $2\cos 4x \cos x = \cos 5x + \cos 3x$, so the right side is $(\cos 5x + \cos 3x)^3$. Setting $y = \cos 3x$ and $z = \cos 5x$, the equation becomes $y^3 + z^3 = (y+z)^3$, and since $(y+z)^3 - y^3 - z^3 = 3yz(y+z)$, it holds exactly when

$$\cos 3x = 0,$$
$$\cos 5x = 0,$$

or

$$\cos 4x \cos x = 0.$$

For $100 < x < 200$ in degrees: $\cos 3x = 0$ gives $x = 150$; $\cos 5x = 0$ gives $x = 126, 162, 198$; $\cos 4x = 0$ gives $x = 112.5, 157.5$; and $\cos x = 0$ gives no solutions in the interval.

The sum is

$$\begin{aligned} &150 + 126 + 162 \\ &\quad + 198 + 112.5 + 157.5 \\ &= 906. \end{aligned}$$

**13.** For each even positive integer $x$, let $g(x)$ denote the greatest power of $2$ that divides $x$. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer $n$, let $S_n = \sum_{k=1}^{2^{n-1}} g(2k)$. Find the greatest integer $n$ less than $1000$ such that $S_n$ is a perfect square.

## Solution:

$S_n$ is the sum of $g$ over the even numbers $2, 4, \ldots, 2^n$. Among these $2^{n-1}$ numbers, exactly $2^{n-1-i}$ are divisible by $2^i$ but not $2^{i+1}$ (so have $g = 2^i$) for each $1 \le i \le n-1$, and the single number $2^n$ has $g = 2^n$. Hence

$$\begin{aligned} S_n &= \sum_{i=1}^{n-1} 2^i \cdot 2^{n-1-i} \\ &\quad + 2^n \\ &= (n-1)2^{n-1} + 2^n \\ &= (n+1)2^{n-1}. \end{aligned}$$

If $n$ is even, then $n+1$ is odd and the exponent $n-1$ is odd, so $S_n$ has an odd number of factors of $2$ and cannot be a perfect square. If $n$ is odd, then $2^{n-1}$ is already a perfect square, so $S_n$ is a perfect square exactly when $n+1$ is.

For odd $n$, the number $n+1$ is even, and the greatest even perfect square at most $1000$ is $900 = 30^2$. Thus the greatest valid $n < 1000$ is $n = 899$.

**14.** A tripod has three legs each of length $5$ feet. When the tripod is set up, the angle between any pair of legs is equal to the angle between any other pair, and the top of the tripod is $4$ feet from the ground. In setting up the tripod, the lower $1$ foot of one leg breaks off. Let $h$ be the height in feet of the top of the tripod from the ground when the broken tripod is set up. Then $h$ can be written in the form $\frac{m}{\sqrt{n}}$, where $m$ and $n$ are positive integers and $n$ is not divisible by the square of any prime. Find $\lfloor m+\sqrt{n}\rfloor$. (The notation $\lfloor x\rfloor$ denotes the greatest integer that is less than or equal to $x$.)

## Solution:

Place the top at $T=(0,0,4)$. Each leg has length $5$, so each foot is $3$ from the origin: $A=(3,0,0), B=\left(-\frac{3}{2},\frac{3\sqrt{3}}{2},0\right), C=\left(-\frac{3}{2},-\frac{3\sqrt{3}}{2},0\right)$. The broken leg has length $4$, so its tip is $A'=\frac{4}{5}(3,0,0)+\frac{1}{5}(0,0,4)=\left(\frac{12}{5},0,\frac{4}{5}\right)$, and the tripod now stands on the plane $A'BC$, with $h$ equal to the distance from $T$ to that plane.

The plane $A'BC$ contains $\overline{BC}$, which is parallel to the $y$-axis and passes through the midpoint $M=\left(-\frac{3}{2},0,0\right)$, so its distance from $T$ can be measured in the $xz$-plane: it is the distance from $(0,4)$ to the line through $\left(-\frac{3}{2},0\right)$ and $\left(\frac{12}{5},\frac{4}{5}\right)$, whose equation is $8x-39z+12=0$. Therefore

$$\begin{aligned} h &= \frac{|8\cdot 0-39\cdot 4+12|}{\sqrt{8^2+39^2}} \\ &= \frac{144}{\sqrt{1585}}. \end{aligned}$$

Here $m=144$ and $n=1585=5\cdot 317$ is squarefree. Since $39^2=1521<1585<1600$, we get $\lfloor 144+\sqrt{1585}\rfloor=144+39=183.$

**15.** Given that a sequence satisfies $x_0 = 0$ and $|x_k| = |x_{k-1} + 3|$ for all integers $k \geq 1$, find the minimum possible value of $|x_1 + x_2 + \cdots + x_{2006}|$.

## Solution:

Squaring the recurrence gives $x_k^2 = (x_{k-1} + 3)^2 = x_{k-1}^2 + 6x_{k-1} + 9$. Summing for $k = 1$ to $2007$ telescopes:

$$x_{2007}^2 = x_0^2 + 6\sum_{k=0}^{2006} x_k + 9 \cdot 2007,$$

so with $x_0 = 0$,

$$\begin{aligned} &x_1 + x_2 + \cdots + x_{2006} \\ &= \frac{x_{2007}^2 - 18063}{6}. \end{aligned}$$

Induction shows each $x_k$ is a multiple of $3$ whose parity matches that of $k$, so $x_{2007}$ is an odd multiple of $3$. To minimize $\left|x_{2007}^2 - 18063\right|$, take the odd multiple of $3$ whose square is nearest $18063$ : that is $\pm 135$, with $135^2 = 18225$, giving $\frac{18225-18063}{6} = 27$ (the neighbors $129$ and $141$ give $237$ and $303$).

This value is attained: take $x_k = 3k$ for $k \leq 45$, and thereafter alternate $x_k = -138$ for even $k$ and $x_k = 135$ for odd $k$; then $x_{2007} = 135$ and the sum is $27$. So the minimum is $27$.

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