2007 AIME II Solutions

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1. A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in AIME and the four digits in 2007. No character may appear in a sequence more times than it appears among the four letters in AIME or the four digits in 2007. A set of plates in which each possible sequence appears exactly once contains N license plates. Find $\frac{N}{10}$.

Solution:

The available characters are the seven distinct symbols A, I, M, E, $2, 0, 7,$ where 0 may be used up to twice and every other character at most once. Sequences using at most one 0 consist of five distinct characters chosen from the seven, in order: $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520.$

Sequences with two 0's: choose the two positions for the 0's in $\binom{5}{2} = 10$ ways, then fill the remaining three positions with distinct characters from the other six in $6 \cdot 5 \cdot 4 = 120$ ways, for 1200 sequences.

Thus $N = 2520 + 1200 = 3720,$ and $\frac{N}{10} = 372.$

2. Find the number of ordered triples (a, b, c) where $a, b,$ and c are positive integers, a is a factor of b, a is a factor of c, and $a + b + c = 100.$

Solution:

Since a divides b and c, it divides $a + b + c = 100$. Write $b = as$ and $c = at$ with $s, t \geq 1$; then $a(1 + s + t) = 100$, so

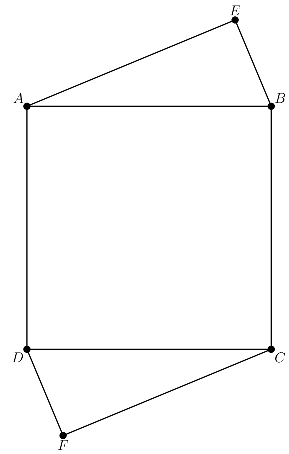
$$s + t = \frac{100}{a} - 1.$$

For positive s and t we need $\frac{100}{a} \geq 3$, so $a \in \{1, 2, 4, 5, 10, 20, 25\}$.

For each such a, the equation $s + t = \frac{100}{a} - 1$ has $\frac{100}{a} - 2$ ordered positive solutions. Summing,

$$\begin{aligned}
&(100 + 50 + 25 + 20 + 10 + 5 + 4) \\
&\quad - 2 \cdot 7 \\
&= 214 - 14 = 200.
\end{aligned}$$

3. Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.



Solution:

Since $5^2 + 12^2 = 13^2$, triangles AEB and CFD are right-angled at E and F, and they are congruent (sides $5, 12, 13$). Extend EA beyond A and FD beyond D until the two lines meet at G.

Then $\angle GAD = 90^\circ - \angle EAB = \angle ABE$, and $\angle GDA = 90^\circ - \angle FDC = \angle DCF = \angle BAE$. These two angles sum to 90°, so $\angle AGD = 90^\circ$, and triangle AGD is congruent to BEA (equal angles and hypotenuse $AD = BA = 13$). Hence $GA = EB = 5$ and $GD = EA = 12$.

Therefore $GE = GA + AE = 5 + 12 = 17$ and $GF = GD + DF = 12 + 5 = 17$, with a right angle between them at G, so $EF^2 = 17^2 + 17^2 = 578.$

4. The workers in a factory produce widgets and whoosits. For each product, production time is constant and identical for all workers, but not necessarily equal for the two products. In one hour, 100 workers can produce 300 widgets and 200 whoosits. In two hours, 60 workers can produce 240 widgets and 300 whoosits. In three hours, 50 workers can produce 150 widgets and m whoosits. Find m.

Solution:

Let a and b be the worker-hours required to make one widget and one whoosit. The three scenarios supply $100, 120,$ and 150 worker-hours, so

$$\begin{aligned} 300a + 200b &= 100, \\ 240a + 300b &= 120, \\ 150a + mb &= 150. \end{aligned}$$

The first two equations simplify to $3a + 2b = 1$ and $4a + 5b = 2,$ giving $a = \frac{1}{7}$ and $b = \frac{2}{7}$. Substituting into the third, $\frac{150}{7} + \frac{2m}{7} = 150,$ so $150 + 2m = 1050$ and $m = 450.$

5. The graph of the equation $9x + 223y = 2007$ is drawn on graph paper with each square representing one unit in each direction. How many of the 1 by 1 graph paper squares have interiors lying entirely below the graph and entirely in the first quadrant?

Solution:

The line meets the axes at $(223, 0)$ and $(0, 9)$, so all qualifying squares lie inside the 223×9 rectangle, which contains $223 \cdot 9 = 2007$ unit squares. Because $\gcd(9, 223) = 1$, the segment passes through no interior lattice point; it crosses 222 interior vertical lines and 8 interior horizontal lines, entering a new square at each crossing, so it passes through the interiors of $223 + 9 - 1 = 231$ squares.

The other $2007 - 231 = 1776$ squares lie entirely above or entirely below the segment. The segment's midpoint $\left(\frac{223}{2}, \frac{9}{2}\right)$ is the center of the rectangle, so rotating by 180° about it swaps the two groups. Hence exactly half of them, 888, lie below the graph.

6. An integer is called *parity-monotonic* if its decimal representation $a_1a_2a_3 \ldots a_k$ satisfies $a_i < a_{i+1}$ if a_i is odd, and $a_i > a_{i+1}$ if a_i is even. How many four-digit parity-monotonic integers are there?

Solution:

A digit a_i may immediately precede $a_{i+1} = d$ exactly when a_i is odd and less than d, or even and greater than d. Checking each d from 0 to 9, this always allows exactly 4 digits: for example, $d = 0$ allows $2, 4, 6, 8$; $d = 4$ allows $1, 3, 6, 8$; $d = 9$ allows $1, 3, 5, 7$. (Raising d by 1 trades odd choices for even ones, keeping the total at 4.) Note that 0 is never an allowed predecessor, since 0 is even but exceeds no digit.

So choose the last digit a_4 in 10 ways, then each of a_3, a_2, a_1 in 4 ways; the leading digit is automatically nonzero. The count is $4^3 \cdot 10 = 640.$

7. Given a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For a certain integer k, there are exactly 70 positive integers $n_1, n_2, \ldots, n_{70}$ such that

$$\begin{aligned} k &= \lfloor \sqrt[3]{n_1} \rfloor = \lfloor \sqrt[3]{n_2} \rfloor \\ &= \cdots = \lfloor \sqrt[3]{n_{70}} \rfloor \end{aligned}$$

and k divides n_i for all i such that $1 \leq i \leq 70$. Find the maximum value of $\frac{n_i}{k}$ for $1 \leq i \leq 70$.

Solution:

The condition $\lfloor \sqrt[3]{n} \rfloor = k$ means $k^3 \leq n < (k+1)^3 = k^3 + 3k^2 + 3k + 1$. The multiples of k in this range are $k \cdot k^2, k(k^2+1), \ldots, k(k^2+3k+3)$, so there are exactly $3k+4$ of them.

Setting $3k + 4 = 70$ gives $k = 22$. The maximum of $\frac{n_i}{k}$ is $k^2 + 3k + 3 = 484 + 66 + 3 = 553$.

8. A rectangular piece of paper measures 4 units by 5 units. Several lines are drawn parallel to the edges of the paper. A rectangle determined by the intersections of some of these lines is called *basic* if (i) all four sides of the rectangle are segments of drawn line segments, and (ii) no segments of drawn lines lie inside the rectangle.

Given that the total length of all lines drawn is exactly 2007 units, let N be the maximum possible number of basic rectangles determined. Find the remainder when N is divided by 1000.

Solution:

Suppose h of the drawn lines have length 4 and v have length 5, so $4h + 5v = 2007$. A basic rectangle is bounded by two adjacent lines in each direction, so the lines determine $(h-1)(v-1)$ basic rectangles. Setting $x = h - 1$ and $y = v - 1$, we must maximize xy subject to $4x + 5y = 1998$.

As a function of x, the product $xy = x \cdot \frac{1998-4x}{5}$ is a downward parabola with vertex at $x = \frac{999}{4} = 249.75$. For y to be an integer we need $4x \equiv 1998 \pmod{5}$, i.e. $x \equiv 2 \pmod{5}$. The nearest candidates are $x = 247$ (giving $y = 202$ and product 49894) and $x = 252$ (giving $y = 198$ and product 49896).

So $N = 49896$, and the remainder upon division by 1000 is 896.

9. Rectangle $ABCD$ is given with $AB = 63$ and $BC = 448$. Points E and F lie on $\overline{AD}$ and $\overline{BC}$ respectively, such that $AE = CF = 84$. The inscribed circle of triangle BEF is tangent to $\overline{EF}$ at point P, and the inscribed circle of triangle DEF is tangent to $\overline{EF}$ at point Q. Find PQ.

Solution:

Place $A = (0, 0), B = (63, 0), C = (63, 448), D = (0, 448)$, so $E = (0, 84)$ and $F = (63, 364)$. Then $BE = DF = \sqrt{63^2 + 84^2} = 21\sqrt{3^2 + 4^2} = 105, BF = DE = 448 - 84 = 364$, and $EF = \sqrt{63^2 + 280^2} = 7\sqrt{9^2 + 40^2} = 287$. In particular triangles BEF and DFE are congruent, with common semiperimeter $s = \frac{105+364+287}{2} = 378$.

In any triangle, the distance from a vertex to the incircle's tangency points on its two sides is the semiperimeter minus the opposite side. In triangle BEF, $EP = s - BF = 378 - 364 = 14$; in triangle DEF, $FQ = s - DE = 378 - 364 = 14$.

Therefore $PQ = EF - EP - FQ = 287 - 14 - 14 = 259$.

10. Let S be a set with six elements. Let $\mathcal{P}$ be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from $\mathcal{P}$. The probability that B is contained in at least one of A or $S - A$ is $\frac{m}{n^r}$, where m, n, and r are positive integers, n is prime, and m and n are relatively prime. Find $m + n + r$. (The set $S - A$ is the set of all elements of S which are not in A.)

Solution:

Fix A with $|A| = k$. There are 2^k subsets $B \subseteq A$ and 2^{6-k} subsets $B \subseteq S - A$, and only the empty set is counted twice, so $2^k + 2^{6-k} - 1$ choices of B succeed. Since there are $\binom{6}{k}$ sets A of size k and 2^6 choices for each of A and B, the probability is

$$\frac{1}{2^{12}} \sum_{k=0}^{6} \binom{6}{k} \left(2^k + 2^{6-k} - 1\right)$$
$$= \frac{2 \cdot 3^6 - 2^6}{2^{12}},$$

using $\sum_k \binom{6}{k} 2^k = \sum_k \binom{6}{k} 2^{6-k} = (1+2)^6 = 3^6$.

This simplifies to $\frac{3^6 - 2^5}{2^{11}} = \frac{697}{2^{11}}$. Since $697 = 17 \cdot 41$ is odd, we take $m = 697$, $n = 2$, $r = 11$, and $m + n + r = 710$.

11. Two long cylindrical tubes of the same length but different diameters lie parallel to each other on a flat surface. The larger tube has radius 72 and rolls along the surface toward the smaller tube, which has radius 24. It rolls over the smaller tube and continues rolling along the flat surface until it comes to rest on the same point of its circumference as it started, having made one complete revolution. If the smaller tube never moves, and the rolling occurs with no slipping, the larger tube ends up a distance x from where it starts. The distance x can be expressed in the form $a\pi + b\sqrt{c}$, where $a, b,$ and c are integers and c is not divisible by the square of any prime. Find $a+b+c$.

Solution:

When the rolling tube touches both the ground and the small tube, the segment between centers has length $72+24=96$ and vertical component $72-24=48$, so it makes a 30° angle with the horizontal. As the big tube rolls over the small one, its center swings along an arc of radius 96 about the small tube's center, from 30° above the horizontal on one side to 30° on the other: a sweep of 120°, advancing the center horizontally by $2 \cdot 96 \cos 30^\circ = 96\sqrt{3}$.

During that sweep, the contact arc on the small tube is $120^\circ \cdot \frac{24}{72} = 40^\circ$ worth of the big tube's circumference, and the sweep itself also rotates the big tube by 120°, so crossing the small tube turns the big tube by 160° in all. To complete exactly one revolution, the remaining $360^\circ - 160^\circ = 200^\circ$ of turning happens rolling on flat ground, where the center advances the rolled distance $\frac{200}{360} \cdot 2\pi \cdot 72 = 80\pi$.

Hence $x = 80\pi + 96\sqrt{3}$, and $a+b+c = 80+96+3 = 179$.

12. The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that

$$\sum_{n=0}^{7} \log_3(x_n) = 308$$

and

$$56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57,$$

find $\log_3(x_{14})$.

Solution:

Every term is a power of 3 and the ratio is a quotient of powers of 3, so $x_n = 3^{a+bn}$ for integers a and b, with $b \geq 1$ since the sequence increases. The first condition gives

$$\sum_{n=0}^{7}(a+bn) = 8a + 28b = 308,$$

i.e.

$$2a + 7b = 77.$$

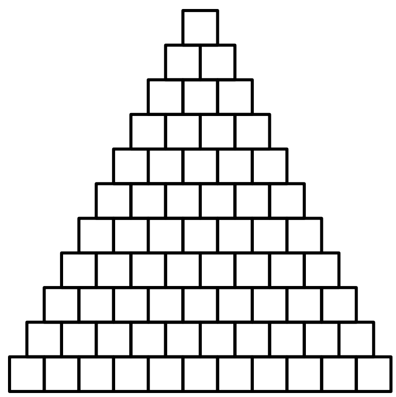
For the second condition, $x_7 = 3^{a+7b}$ is the largest term, and

$$\begin{aligned} x_7 &< \sum_{n=0}^{7} x_n \\ &< x_7\left(1 + \tfrac{1}{3} + \tfrac{1}{9} + \cdots\right) \\ &= \tfrac{3}{2}\, x_7 < 3x_7, \end{aligned}$$

so $\log_3\left(\sum x_n\right)$ lies strictly between $a+7b$ and $a+7b+1$. The given bounds then force $a + 7b = 56$.

Subtracting from $2a + 7b = 77$ yields $a = 21$, then $b = 5$. Therefore $\log_3(x_{14}) = a + 14b = 21 + 70 = 91$.

13. A triangular array of squares has one square in the first row, two in the second, and, in general, k squares in the kth row for $1 \leq k \leq 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below, as illustrated in the figure. In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?



Solution:

Label the bottom-row entries $x_0, x_1, \ldots, x_{10}$. Since each square is the sum of the two below it, the contributions accumulate with Pascal's-triangle weights: the top square equals $\sum_{i=0}^{10} \binom{10}{i} x_i$.

Modulo 3, direct checking (or Lucas' theorem with $10 = 101_3$) shows $\binom{10}{i} \equiv 0$ for $2 \leq i \leq 8$, while $\binom{10}{0} = \binom{10}{10} = 1$ and $\binom{10}{1} = \binom{10}{9} = 10 \equiv 1$. So the top square is a multiple of 3 exactly when $x_0 + x_1 + x_9 + x_{10} \equiv 0 \pmod{3}$.

For entries equal to 0 or 1, this sum is 0 or 3 : either all four are 0 (one way) or exactly three are 1 (four ways), for 5 choices. The remaining seven entries $x_2, \ldots, x_8$ are free, so the count is $5 \cdot 2^7 = 640$.

14. Let $f(x)$ be a polynomial with real coefficients such that $f(0) = 1, f(2) + f(3) = 125$, and for all x, $f(x)f(2x^2) = f(2x^3 + x)$. Find $f(5)$.

Solution:

If f has degree m and leading coefficient a, the leading coefficients of the two sides of $f(x)f(2x^2) = f(2x^3 + x)$ are $a^2 2^m$ and $a2^m$, so $a = 1$. The equation also shows that whenever λ is a root, $2\lambda^3 + \lambda$ is a root as well.

If some root had $|\lambda| > 1$, then $|2\lambda^3 + \lambda| \geq 2|\lambda|^3 - |\lambda| > |\lambda|$, and iterating would produce infinitely many distinct roots — impossible. Since f is monic with $f(0) = 1$, the product of the roots has modulus 1, so no root can have modulus less than 1 either: every root satisfies $|\lambda| = 1$. Then $2\lambda^3 + \lambda$ must also have modulus 1, so $|2\lambda^2 + 1| = 1$. Writing $\lambda^2 = \cos\theta + i\sin\theta$, we get $(2\cos\theta + 1)^2 + 4\sin^2\theta = 1$, which simplifies to $\cos\theta = -1$, so $\lambda^2 = -1$.

Thus every root is $\pm i$, and real coefficients pair them up: $f(x) = (x^2 + 1)^n$. The condition $f(2) + f(3) = 5^n + 10^n = 125$ gives $n = 2$, so $f(5) = 26^2 = 676$.

15. Four circles $\omega, \omega_A, \omega_B$, and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B, and ω_C. If the sides of triangle ABC are $13, 14,$ and $15,$ the radius of ω can be represented in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Let x be the common radius, and let O_A, O_B, O_C be the centers of $\omega_A, \omega_B, \omega_C$. Each is at distance x from two sides of the triangle, so each lies on an angle bisector, and the sides of triangle $O_AO_BO_C$ are parallel to those of ABC at distance x. Hence $O_AO_BO_C$ is the image of ABC under the homothety centered at the incenter I with ratio $\frac{r-x}{r}$, where r is the inradius; in particular its circumradius is $R \cdot \frac{r-x}{r}$, where R is the circumradius of ABC.

The center of ω is at distance $2x$ from each of O_A, O_B, O_C (externally tangent equal circles), so it is the circumcenter of $O_AO_BO_C$ and

$$2x = R \cdot \frac{r-x}{r}.$$

For the 13-14-15 triangle, $s=21$ and Heron's formula gives area $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, so $r = \frac{84}{21} = 4$ and $R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}$.
Then $2x = \frac{65}{8} \cdot \frac{4-x}{4}$ gives $64x = 260 - 65x$, so $x = \frac{260}{129}$. Since $129 = 3 \cdot 43$ shares no factor with 260, the answer is $m+n = 260+129 = 389$.

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