

2009 AIME I

Time limit: 180 minutes

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1. Call a **3**-digit number *geometric* if it has **3** distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.
2. There is a complex number z with imaginary part **164** and a positive integer n such that

$$\frac{z}{z + n} = 4i.$$

Find n .

3. A coin that comes up heads with probability $p > 0$ and tails with probability $1 - p > 0$ independently on each flip is flipped eight times. Suppose the probability of three heads and five tails is equal to $\frac{1}{25}$ of the probability of five heads and three tails. Let $p = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$, and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.
5. Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C . Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP .
6. How many positive integers N less than 1000 are there such that the equation $x^{\lfloor x \rfloor} = N$ has a solution for x ? (The notation $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x .)
7. The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)} - 1 = \frac{1}{n+\frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k .

8. Let $S = \{2^0, 2^1, 2^2, \dots, 2^{10}\}$. Consider all possible positive differences of pairs of elements of S . Let N be the sum of all of these differences. Find the remainder when N is divided by 1000.
9. A game show offers a contestant three prizes A, B and C, each of which is worth a whole number of dollars from \$1 to \$9999 inclusive. The contestant wins the prizes by correctly guessing the price of each prize in the order A, B, C. As a hint, the digits of the three prices are given. On a particular day, the digits given were 1, 1, 1, 1, 3, 3, 3. Find the total number of possible guesses for all three prizes consistent with the hint.
10. The Annual Interplanetary Mathematics Examination (AIME) is written by a committee of five Martians, five Venusians, and five Earthlings. At meetings, committee members sit at a round table with chairs numbered from 1 to 15 in clockwise order. Committee rules state that a Martian must occupy chair 1 and an Earthling must occupy chair 15. Furthermore, no Earthling can sit immediately to the left of a Martian, no Martian can sit immediately to the left of a Venusian, and no Venusian can sit immediately to the left of an Earthling. The number of possible seating arrangements for the committee is $N \cdot (5!)^3$. Find N .

11. Consider the set of all triangles OPQ where O is the origin and P and Q are distinct points in the plane with nonnegative integer coordinates (x, y) such that $41x + y = 2009$. Find the number of such distinct triangles whose area is a positive integer.
12. In right $\triangle ABC$ with hypotenuse $\overline{AB}$, $AC = 12$, $BC = 35$, and $\overline{CD}$ is the altitude to $\overline{AB}$. Let ω be the circle having $\overline{CD}$ as a diameter. Let I be a point outside $\triangle ABC$ such that $\overline{AI}$ and $\overline{BI}$ are both tangent to circle ω . The ratio of the perimeter of $\triangle ABI$ to the length AB can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
13. The terms of the sequence (a_i) defined by $a_{n+2} = \frac{a_n + 2009}{1 + a_{n+1}}$ for $n \geq 1$ are positive integers. Find the minimum possible value of $a_1 + a_2$.
14. For $t = 1, 2, 3, 4$, define $S_t = \sum_{i=1}^{350} a_i^t$, where $a_i \in \{1, 2, 3, 4\}$. If $S_1 = 513$ and $S_4 = 4745$, find the minimum possible value for S_2 .

15. In triangle ABC , $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD , respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and Q . The maximum possible area of $\triangle BPQ$ can be expressed in the form $a - b\sqrt{c}$, where a , b , and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

Solutions: <https://live.poshenloh.com/past-contests/aime/2009I/solutions>

