2011 AIME I Solutions

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1. Jar A contains four liters of a solution that is 45% acid. Jar B contains five liters of a solution that is 48% acid. Jar C contains one liter of a solution that is $k\%$ acid. From jar C, $\frac{m}{n}$ liters of the solution is added to jar A, and the remainder of the solution in jar C is added to jar B. At the end both jar A and jar B contain solutions that are 50% acid. Given that m and n are relatively prime positive integers, find $k+m+n$.

Solution:

If all three jars were combined, the result would be 10 liters of 50% acid, since both final jars are 50% acid. The total acid is therefore 5 liters, so $4(0.45)+5(0.48)+0.01k=5$, which gives $k=80$.

Now let x be the number of liters poured from jar C into jar A. Jar A then holds $4+x$ liters containing $1.8+0.8x$ liters of acid, so $1.8+0.8x=0.5(4+x)$, giving $0.3x=0.2$, so $x=\frac{2}{3}$.

Thus $m+n=2+3=5$, and $k+m+n=80+5=85$.

2. In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9, DF = 8, \overline{BE} \parallel \overline{DF}, \overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. The length EF can be expressed in the form $m\sqrt{n} - p$, where m, n, and p are positive integers and n is not divisible by the square of any prime. Find $m + n + p$.

Solution:

Place $D = (0, 0), C = (12, 0), B = (12, 10), A = (0, 10)$. Since $\overline{BE} \parallel \overline{DF}$, there is a unit vector (u, v) with $u, v > 0$ such that $E = B - 9(u, v)$ and $F = D + 8(u, v)$: line BE heads down and to the left so that it can cross $\overline{AD}$, while $\overline{DF}$ points up and to the right into the rectangle.

Because $\overline{EF} \parallel \overline{AB}$ is horizontal, E and F have equal heights: $10 - 9v = 8v$, so $v = \frac{10}{17}$ and $u = \sqrt{1 - v^2} = \frac{3\sqrt{21}}{17}$.

Then E and F have x-coordinates $12 - 9u$ and $8u$, so $EF = |12 - 17u| = |12 - 3\sqrt{21}| = 3\sqrt{21} - 12$, since $3\sqrt{21} > 12$. Thus $m + n + p = 3 + 21 + 12 = 36$.

3. Let L be the line with slope $\frac{5}{12}$ that contains the point $A = (24, -1)$, and let M be the line perpendicular to line L that contains the point $B = (5, 6)$. The original coordinate axes are erased, and line L is made the x-axis and line M the y-axis. In the new coordinate system, point A is on the positive x-axis, and point B is on the positive y-axis. The point P with coordinates $(-14, 27)$ in the original system has coordinates (α, β) in the new coordinate system. Find $\alpha + \beta$.

Solution:

Line L is $5x - 12y - 132 = 0$ and line M is $12x + 5y - 90 = 0$. The new x-coordinate of a point is its signed distance to line M, counted positive on the side containing A, and the new y-coordinate is its signed distance to line L, positive on the side containing B.

Substituting $P = (-14, 27)$ into $12x + 5y - 90$ gives $-168 + 135 - 90 = -123$, while A gives $193 > 0$; dividing by $\sqrt{12^2 + 5^2} = 13$, we get $\alpha = -\frac{123}{13}$. Substituting P into $5x - 12y - 132$ gives $-70 - 324 - 132 = -526$, and B gives -179, so P lies on the same side of L as B and $\beta = \frac{526}{13}$.

Therefore $\alpha + \beta = \frac{-123+526}{13} = \frac{403}{13} = 31.$

4. In triangle $ABC, AB = 125, AC = 117,$ and $BC = 120.$ The angle bisector of angle A intersects $\overline{BC}$ at point $L,$ and the angle bisector of angle B intersects $\overline{AC}$ at point $K.$ Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL},$ respectively. Find $MN.$

Solution:

Extend $\overline{CM}$ and $\overline{CN}$ to meet $\overline{AB}$ at P and $Q,$ respectively. In triangle $BCP,$ the segment BM is both an angle bisector and an altitude, so the triangle is isosceles with $BP = BC = 120,$ and M is the midpoint of $\overline{CP}.$ Similarly, triangle ACQ is isosceles with $AQ = AC = 117,$ and N is the midpoint of $\overline{CQ}.$

Hence $\overline{MN}$ is a midline of triangle $CPQ,$ so $MN = \frac{PQ}{2}.$ Since

$$\begin{aligned} PQ &= BP + AQ - AB \\ &= 120 + 117 - 125 \\ &= 112, \end{aligned}$$

we conclude $MN = 56.$

5. The vertices of a regular nonagon (9-sided polygon) are to be labeled with the digits 1 through 9 in such a way that the sum of the numbers on every three consecutive vertices is a multiple of 3. Two acceptable arrangements are considered to be indistinguishable if one can be obtained from the other by rotating the nonagon in the plane. Find the number of distinguishable acceptable arrangements.

Solution:

Two overlapping triples of consecutive vertices share two labels, so their sums differ by labels three positions apart. Since all triple sums are multiples of 3, labels three apart are congruent mod 3. Thus the position classes $\{1,4,7\},\{2,5,8\},\{3,6,9\}$ each carry a single residue class of digits, and the digits 1 through 9 consist of exactly three digits from each residue class mod 3.

Every triple of consecutive positions then contains one digit from each residue class, with sum $\equiv 0+1+2 \equiv 0 \pmod 3$, so all $3!$ assignments of residue classes to position classes are acceptable, and within each position class the three digits can be arranged in $3!$ ways. That gives $3! \cdot (3!)^3 = 6^4 = 1296$ acceptable labelings.

Because the digits are distinct, no nontrivial rotation fixes a labeling, so the 1296 labelings split into rotation classes of size 9, giving $\frac{1296}{9} = 144$ distinguishable arrangements.

6. Suppose that a parabola has vertex $\left(\frac{1}{4}, -\frac{9}{8}\right)$ and equation $y = ax^2 + bx + c$, where $a > 0$ and $a + b + c$ is an integer. The minimum possible value of a can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:

In vertex form the parabola is $y = a\left(x - \frac{1}{4}\right)^2 - \frac{9}{8}$. Since $a + b + c$ equals the value of y at $x = 1$,

$$\begin{aligned} a + b + c &= a\left(\frac{3}{4}\right)^2 - \frac{9}{8} \\ &= \frac{9(a-2)}{16}. \end{aligned}$$

If this equals the integer n, then $a = 2 + \frac{16n}{9}$. The condition $a > 0$ requires $16n > -18$, that is $n \geq -1$, and a is smallest when $n = -1$, giving $a = 2 - \frac{16}{9} = \frac{2}{9}$.

Thus $p + q = 2 + 9 = 11$.

7. Find the number of positive integers m for which there exist nonnegative integers x_0, $x_1, \ldots, x_{2011}$, such that

$$m^{x_0} = \sum_{k=1}^{2011} m^{x_k}.$$

Solution:

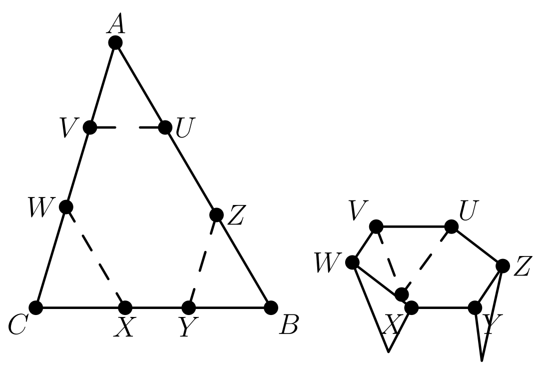
The value $m = 1$ fails, since the right side would be 2011 while the left side is 1. For $m \geq 2$, reduce mod $m - 1$: every power of m is $\equiv 1$, so the equation forces $1 \equiv 2011 \pmod{m-1}$, that is, $m - 1$ divides 2010.

Conversely, suppose $2010 = (m-1)n$. Take $x_0 = n$, let m of the x_k equal 0, and for each $r = 1, 2, \ldots, n-1$ let $m - 1$ of the x_k equal r. This uses $m + (m-1)(n-1)$ $= n(m-1) + 1 = 2011$ terms, and the sum telescopes:

$$\begin{aligned} m & \\ & + (m-1) \\ & \cdot (m + m^2 + \cdots + m^{n-1}) \\ & = m + (m^n - m) \\ & = m^n = m^{x_0}. \end{aligned}$$

So the equation is solvable exactly when $m - 1$ divides $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, which has $2^4 = 16$ divisors. There are 16 such m.

8. In $\triangle ABC, BC = 23, CA = 27,$ and $AB = 30.$ Points V and W are on $\overline{AC}$ with V on $\overline{AW}$, points X and Y are on $\overline{BC}$ with X on $\overline{CY}$, and points Z and U are on $\overline{AB}$ with Z on $\overline{BU}$. In addition, the points are positioned so that $\overline{UV} \parallel \overline{BC}, \overline{WX} \parallel \overline{AB},$ and $\overline{YZ} \parallel \overline{CA}.$ Right angle folds are then made along $\overline{UV}, \overline{WX}$, and $\overline{YZ}$. The resulting figure is placed on a level floor to make a table with triangular legs. Let h be the maximum possible height of a table constructed from $\triangle ABC$ whose top is parallel to the floor. Then h can be written in the form $\frac{k\sqrt{m}}{n}$, where k and n are relatively prime positive integers and m is a positive integer that is not divisible by the square of any prime. Find $k + m + n.$



Solution:

Write $a = BC = 23, b = CA = 27, c = AB = 30,$ and let K be the area of $\triangle ABC.$ By Heron's formula with semiperimeter 40, $K = \sqrt{40 \cdot 17 \cdot 13 \cdot 10} = 20\sqrt{221}.$ When the corner at a vertex is folded down at a right angle, the flap hangs to a depth equal to the distance from that vertex to the fold line, so for a level tabletop of height h, each fold line must lie at distance h from its vertex.

The flap at A is similar to $\triangle ABC$ with ratio $\frac{h}{\frac{2K}{a}} = \frac{ha}{2K}$ (dividing h by the distance from A to $\overline{BC}$), so it uses up $AU = c \cdot \frac{ha}{2K}$ of side $\overline{AB}$; likewise the flap at B uses $BZ = c \cdot \frac{hb}{2K}$ of the same side. The two folds fit without crossing exactly when $AU + BZ \leq c,$ that is, $h(a+b) \leq 2K.$ The other two sides give $h(b+c) \leq 2K$ and $h(c+a) \leq 2K.$

The binding constraint comes from the largest sum, $b + c = 57,$ so the maximum height is

$$h = \frac{2K}{57} = \frac{40\sqrt{221}}{57},$$

and $k + m + n = 40 + 221 + 57 = 318$.

9. Suppose x is in the interval $\left[0, \frac{\pi}{2}\right]$ and $\log_{24 \sin x}(24 \cos x) = \frac{3}{2}$. Find $24 \cot^2 x$.

Solution:

In exponential form the equation says $(24 \sin x)^{\frac{3}{2}} = 24 \cos x$. Squaring gives $24^3 \sin^3 x = 24^2 \cos^2 x$, so $\cos^2 x = 24 \sin^3 x$.

Writing $s = \sin x$ and using $\cos^2 x = 1 - s^2$, we get $24s^3 + s^2 - 1 = 0$, which factors as $(3s - 1)(8s^2 + 3s + 1) = 0$. The quadratic factor has negative discriminant, so $\sin x = \frac{1}{3}$.

Then

$$\begin{aligned} 24 \cot^2 x &= 24 \cdot \frac{1 - \sin^2 x}{\sin^2 x} \\ &= 24 \cdot \frac{\frac{8}{9}}{\frac{1}{9}} \\ &= 24 \cdot 8 = 192. \end{aligned}$$

10. The probability that a set of three distinct vertices chosen at random from among the vertices of a regular n-gon determine an obtuse triangle is $\frac{93}{125}$. Find the sum of all possible values of n.

Solution:

By the inscribed angle theorem, an inscribed triangle is obtuse exactly when its three vertices lie strictly within some semicircle. Count obtuse triangles by their "first" vertex, the vertex from which the other two are reached going clockwise within half the circle. If $n = 2k$, the open semicircle clockwise of a vertex contains $k-1$ vertices, giving $n\binom{k-1}{2}$ obtuse triangles; if $n = 2k+1$, it contains k vertices, giving $n\binom{k}{2}$.

For $n = 2k$ the probability is

$$\frac{2k\binom{k-1}{2}}{\binom{2k}{3}} = \frac{3(k-2)}{2(2k-1)} = \frac{93}{125},$$

so $375(k-2) = 186(2k-1)$, giving $3k = 564$, $k = 188$, and $n = 376$.

For $n = 2k+1$ the probability is $\frac{3(k-1)}{2(2k-1)} = \frac{93}{125}$, so $375(k-1) = 186(2k-1)$, giving $3k = 189$, $k = 63$, and $n = 127$. The sum of all possible values is $376 + 127 = 503$.

11. Let R be the set of all possible remainders when a number of the form 2^n, n a nonnegative integer, is divided by 1000. Let S be the sum of the elements in R. Find the remainder when S is divided by 1000.

Solution:

The remainders $2^0 = 1, 2^1 = 2$, and $2^2 = 4$ occur, and for $n \geq 3$ every 2^n is divisible by 8. Modulo 125 the powers of 2 for $n \geq 3$ repeat with period 100, so R consists of 1, $2, 4$, and the 100 distinct remainders of $2^3, 2^4, \ldots, 2^{102}$.

The key fact is $2^{50} \equiv -1 \pmod{125}$: indeed $2^{50} + 1 = (2^{10} + 1) \cdot (2^{40} - 2^{30} + 2^{20} - 2^{10} + 1)$, where $2^{10} + 1 = 1025$ is divisible by 25 and the second factor is $\equiv 1 + 1 + 1 + 1 + 1 \equiv 0 \pmod 5$ because $2^{10} \equiv -1 \pmod 5$. Hence for $n \geq 3$, the sum $2^{n+50} + 2^n$ is divisible by 125 and by 8, so by 1000.

Pairing each remainder in the cycle with the one 50 steps later therefore gives 50 pairs of distinct remainders, each pair summing to exactly 1000, so those 100 remainders contribute a multiple of 1000 to S. Thus $S \equiv 1 + 2 + 4 = 7 \pmod{1000}$.

12. Six men and some number of women stand in a line in random order. Let p be the probability that a group of at least four men stand together in the line, given that every man stands next to at least one other man. Find the least number of women in the line such that p does not exceed 1 percent.

Solution:

Let n be the number of women; only the pattern of men's and women's positions matters. If every man stands next to another man, the men form maximal blocks whose sizes, in order, are $2+2+2, 2+4, 4+2, 3+3,$ or $6.$ A pattern with j blocks amounts to choosing j of the $n+1$ gaps determined by the women, so there are $\binom{n+1}{3}$ patterns for $2+2+2,$ $\binom{n+1}{2}$ for each of the three two-block orders, and $n+1$ for a single block.

At least four men stand together in the orders $2+4, 4+2,$ and $6,$ so

$$p = \frac{2\binom{n+1}{2} + (n+1)}{\binom{n+1}{3} + 3\binom{n+1}{2} + (n+1)} = \frac{(n+1)^2}{\frac{(n+1)(n^2+8n+6)}{6}} = \frac{6(n+1)}{n^2+8n+6}.$$

The condition $p \le \frac{1}{100}$ becomes $f(n) = n^2 - 592n - 594 \ge 0.$ Since $f(593) = 593 - 594 = -1 < 0$ and $f(594) = 2 \cdot 594 - 594 = 594 > 0,$ the least number of women is $594.$

13. A cube with side length 10 is suspended above a plane. The vertex closest to the plane is labeled A. The three vertices adjacent to vertex A are at heights $10, 11,$ and 12 above the plane. The distance from vertex A to the plane can be expressed as $\frac{r-\sqrt{s}}{t}$, where $r, s,$ and t are positive integers. Find $r+s+t$.

Solution:

Let h be the height of A, and let e_1, e_2, e_3 be unit vectors along the three mutually perpendicular edges at A. If u is the upward unit normal of the plane, the height of the vertex along edge i is $h+10(e_i \cdot u)$, so $10(e_1 \cdot u) = 10-h, 10(e_2 \cdot u) = 11-h,$ and $10(e_3 \cdot u) = 12-h$. Because e_1, e_2, e_3 form an orthonormal basis, $(e_1 \cdot u)^2 + (e_2 \cdot u)^2 + (e_3 \cdot u)^2 = 1$.

Therefore

$$\begin{aligned}(10-h)^2 + (11-h)^2 \\ + (12-h)^2 = 100,\end{aligned}$$

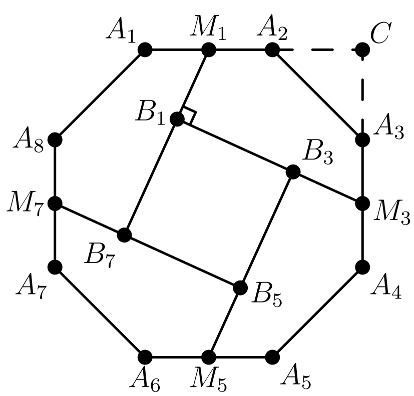
which simplifies to $3h^2 - 66h + 265 = 0$, so $h = \frac{33 \pm \sqrt{33^2 - 3 \cdot 265}}{3} = \frac{33 \pm \sqrt{294}}{3}$.

Since A is the closest vertex to the plane, $h < 10$, forcing $h = \frac{33-\sqrt{294}}{3}$, and $r+s+t = 33+294+3 = 330$.

14. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let M_1, M_3, M_5, and M_7 be the midpoints of sides $\overline{A_1A_2}, \overline{A_3A_4}, \overline{A_5A_6}$, and $\overline{A_7A_8}$, respectively. For $i = 1, 3, 5, 7$, ray R_i is constructed from M_i towards the interior of the octagon such that $R_1 \perp R_3$, $R_3 \perp R_5, R_5 \perp R_7$, and $R_7 \perp R_1$. Pairs of rays R_1 and R_3, R_3 and R_5, R_5 and R_7, and R_7 and R_1 meet at B_1, B_3, B_5, and B_7, respectively. If $B_1B_3 = A_1A_2$, then $\cos 2\angle A_3M_3B_1$ can be written in the form $m - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Solution:

Scale so that $A_1A_2 = 2$. A 90° rotation about the center carries the whole configuration to itself, so $B_1B_3B_5B_7$ is a square and the distances $a = M_iB_i$ and $b = M_iB_{i-2}$ do not depend on i. Both B_3 and B_1 lie on ray R_3 (at distances a and b from M_3), so $b - a = B_1B_3 = 2$. Also, $R_1 \perp R_3$ makes triangle $M_1B_1M_3$ right-angled at B_1, so $a^2 + b^2 = M_1M_3^2$.



Lines A_1A_2 and A_3A_4 meet at a point C at a right angle, and triangle A_2CA_3 is an isosceles right triangle with legs $A_2C = A_3C = \sqrt{2}$, so $M_1C = M_3C = 1 + \sqrt{2}$ and $a^2 + b^2 = M_1M_3^2 = 2(1 + \sqrt{2})^2$. Then

$$\begin{aligned}(a+b)^2 &= 2(a^2+b^2) - (b-a)^2 \\ &= 4(1+\sqrt{2})^2 - 4 \\ &= 8 + 8\sqrt{2}.\end{aligned}$$

Since triangle M_1CM_3 is an isosceles right triangle and A_3 lies on segment $\overline{M_3C}$, we have $\angle A_3M_3M_1 = 45^\circ$, while $\tan \angle M_1M_3B_1 = \frac{a}{b}$ from the right triangle. The tangent addition formula gives

$$\begin{aligned}\tan\angle A_3M_3B_1 &= \frac{1+\frac{a}{b}}{1-\frac{a}{b}}\\ &= \frac{a+b}{b-a},\end{aligned}$$

so

$$\begin{aligned}\tan^2\angle A_3M_3B_1 &= \frac{8+8\sqrt{2}}{4}\\ &= 2+2\sqrt{2}.\end{aligned}$$

Therefore

$$\begin{aligned}&\cos 2\angle A_3M_3B_1\\ &\quad= \tfrac{1-\tan^2\angle A_3M_3B_1}{1+\tan^2\angle A_3M_3B_1}\\ &\quad= \frac{-1-2\sqrt{2}}{3+2\sqrt{2}}\\ &\quad= -(1+2\sqrt{2})(3-2\sqrt{2})\\ &\quad= 5-4\sqrt{2}\\ &\quad= 5-\sqrt{32},\end{aligned}$$

and $m+n=5+32=37.$

15. For some integer m, the polynomial $x^3 - 2011x + m$ has the three integer roots $a, b,$ and c. Find $|a| + |b| + |c|$.

Solution:

By Vieta's formulas, $a + b + c = 0$ and $ab + bc + ca = -2011$. Negating all three roots replaces m by $-m$, so we may assume two roots, say $a \geq b$, are nonnegative. Substituting $c = -(a + b)$ into the second equation gives $ab - (a + b)^2 = -2011$, that is,

$$a^2 + ab + b^2 = 2011.$$

Multiplying by 4 and completing the square, $(2a + b)^2 + 3b^2 = 8044$. Since $a \geq b \geq 0$, we have $3b^2 \leq a^2 + ab + b^2 = 2011$, so $0 \leq b \leq 25$. Checking these values, $8044 - 3b^2$ is a perfect square only for $b = 10$, where $8044 - 300 = 7744 = 88^2$.

Then $2a + b = 88$ gives $a = 39$, and $c = -(a + b) = -49$. (Indeed $39 \cdot 10 - 49^2 = -2011$.) Therefore $|a| + |b| + |c| = 39 + 10 + 49 = 98$.

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