2011 AIME II

Time limit: 180 minutes

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1. Gary purchased a large beverage, but drank only $\frac{m}{n}$ of this beverage, where m and n are relatively prime positive integers. If Gary had purchased only half as much and drunk twice as much, he would have wasted only $\frac{2}{9}$ as much beverage. Find $m + n$.

2. On square $ABCD$, point E lies on side $\overline{AD}$ and point F lies on side $\overline{BC}$, so that $BE = EF = FD = 30$. Find the area of square $ABCD$.

3. The degree measures of the angles of a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.

4. In triangle $ABC, AB = \frac{20}{11}AC$. The angle bisector of angle A intersects $\overline{BC}$ at point D, and point M is the midpoint of $\overline{AD}$. Let P be the point of the intersection of $\overline{AC}$ and line BM. The ratio of CP to PA can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5. The sum of the first 2011 terms of a geometric series is 200. The sum of the first 4022 terms of the same series is 380. Find the sum of the first 6033 terms of the series.

6. Define an ordered quadruple of integers (a, b, c, d) to be *interesting* if $1 \le a < b < c < d \le 10$ and $a+d > b+c$. How many interesting ordered quadruples are there?

7. Ed has five identical green marbles, and a large supply of identical red marbles. He arranges the green marbles and some of the red ones in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves equals the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is GGRRRGGRG. Let m be the maximum number of red marbles for which such an arrangement is possible, and let N be the number of ways in which Ed can arrange the $m+5$ marbles to satisfy the requirement. Find the remainder when N is divided by 1000.

8. Let $z_1, z_2, z_3, \ldots, z_{12}$ be the 12 zeroes of the polynomial $z^{12} - 2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $\sum_{j=1}^{12} w_j$ can be written as $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

9. Let $x_1, x_2, \ldots, x_6$ be nonnegative real numbers such that $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, and $x_1x_3x_5 + x_2x_4x_6 \geq \frac{1}{540}$. Let p and q be positive relatively prime integers such that $\frac{p}{q}$ is the maximum possible value of

$$\begin{aligned} & x_1x_2x_3 + x_2x_3x_4 \\ & \quad + x_3x_4x_5 + x_4x_5x_6 \\ & \quad + x_5x_6x_1 + x_6x_1x_2. \end{aligned}$$

Find $p + q$.

10. A circle with center O has radius 25. Chord $\overline{AB}$ of length 30 and chord $\overline{CD}$ of length 14 intersect at point P. The distance between the midpoints of the two chords is 12. The quantity OP^2 can be represented as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

11. Let M_n be the $n \times n$ matrix with entries as follows: for $1 \leq i \leq n, m_{i,i} = 10$; for $1 \leq i \leq n-1, m_{i+1,i} = m_{i,i+1} = 3$; all other entries in M_n are zero. Let D_n be the determinant of matrix M_n. Then $\sum_{n=1}^{\infty} \frac{1}{8D_n+1}$ can be represented as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Note: The determinant of the 1×1 matrix $[a]$ is a, and the determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$; for $n \geq 2$, the determinant of an $n \times n$ matrix with first row or first column $a_1\, a_2\, a_3 \ldots a_n$ is equal to $a_1C_1 - a_2C_2 + a_3C_3 - \cdots + (-1)^{n+1}a_nC_n$, where C_i is the determinant of the $(n-1) \times (n-1)$ matrix formed by eliminating the row and column containing a_i.

12. Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

13. Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP, respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120^\circ$, then $AP = \sqrt{a} + \sqrt{b}$, where a and b are positive integers. Find $a+b$.

14. There are N permutations $(a_1, a_2, \dots, a_{30})$ of $1, 2, \dots, 30$ such that for $m \in \{2, 3, 5\}$, m divides $a_{n+m} - a_n$ for all integers n with $1 \le n < n + m \le 30$. Find the remainder when N is divided by 1000.

15. Let $P(x) = x^2 - 3x - 9$. A real number x is chosen at random from the interval $5 \le x \le 15$. The probability that $\left\lfloor \sqrt{P(x)} \right\rfloor = \sqrt{P(\lfloor x \rfloor)}$ is equal to $\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}-d}{e}$, where $a, b, c, d,$ and e are positive integers. Find $a + b + c + d + e$.

Solutions: https://live.poshenloh.com/past-contests/aime/2011II/solutions