2017 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Fifteen distinct points are designated on $\triangle ABC$: the 3 vertices A, B, and C; 3 other points on side $\overline{AB}$; 4 other points on side $\overline{BC}$; and 5 other points on side $\overline{CA}$. Find the number of triangles with positive area whose vertices are among these 15 points.

Solution:

There are $\binom{15}{3} = 455$ ways to choose 3 of the points. A choice fails to give a triangle of positive area exactly when the 3 points are collinear, which happens only when all three lie on one side of the triangle. Including its endpoints, side $\overline{AB}$ contains 5 points, $\overline{BC}$ contains 6, and $\overline{CA}$ contains 7, giving $\binom{5}{3} + \binom{6}{3} + \binom{7}{3} = 10 + 20 + 35 = 65$ collinear triples.

The number of triangles is $455 - 65 = 390$.

2. When each of $702, 787,$ and 855 is divided by the positive integer $m,$ the remainder is always the positive integer $r.$ When each of $412, 722,$ and 815 is divided by the positive integer $n,$ the remainder is always the positive integer $s \neq r.$ Find $m+n+r+s.$

Solution:

Numbers leaving equal remainders upon division by m differ by multiples of $m,$ so m divides both $787-702=85$ and $855-787=68.$ Since $\gcd(85,68)=17$ and m must exceed the positive remainder $r,$ we get $m=17,$ and $r=702-41\cdot 17=5.$

Similarly n divides both $722-412=310$ and $815-722=93,$ and $\gcd(310,93)=31,$ so $n=31$ and $s=412-13\cdot 31=9,$ which indeed differs from $r.$

The requested sum is $17+31+5+9=62.$

3. For a positive integer n, let d_n be the units digit of $1 + 2 + 3 + \cdots + n$. Find the remainder when

$$\sum_{n=1}^{2017} d_n$$

is divided by 1000.

Solution:

Here d_n is the units digit of the triangular number $\frac{n(n+1)}{2}$. Since $\frac{(n+20)(n+21)}{2} - \frac{n(n+1)}{2} = 20n + 210$ is a multiple of 10, the sequence d_n is periodic with period 20. Computing one period,

$$\begin{aligned}&(d_1, d_2, \ldots, d_{20})\\ &\quad = (1, 3, 6, 0, 5, 1, 8, 6, 5, 5, 6, 8, 1, 5, 0, 6, 3, 1, 0, 0),\end{aligned}$$

which sums to 70.

Since $2017 = 100 \cdot 20 + 17$, the total is $100 \cdot 70$ plus the first 17 terms of the period, which sum to $70 - (1 + 0 + 0) = 69$. The sum is 7069, so the remainder is 69.

4. A pyramid has a triangular base with side lengths $20, 20$, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. The volume of the pyramid is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Solution:

Since the apex is equidistant from all three base vertices, its foot is the circumcenter of the base. The base is isosceles with sides $20, 20, 24$: its altitude to the side of length 24 is $\sqrt{20^2-12^2}=16$, so its area is $K=\frac{1}{2}\cdot 24\cdot 16=192$, and its circumradius is

$$R=\frac{abc}{4K}=\frac{20\cdot 20\cdot 24}{4\cdot 192}=\frac{25}{2}.$$

The height of the pyramid is $\sqrt{25^2-\left(\frac{25}{2}\right)^2}=\frac{25\sqrt{3}}{2}$, so the volume is $\frac{1}{3}\cdot 192\cdot\frac{25\sqrt{3}}{2}=800\sqrt{3}$. Then $m+n=800+3=803$.

5. A rational number written in base eight is $\underline{a}\,\underline{b}.\underline{c}\,\underline{d}$, where all digits are nonzero. The same number in base twelve is $\underline{b}\,\underline{b}.\underline{b}\,\underline{a}$. Find the base-ten number $\underline{a}\,\underline{b}\,\underline{c}$.

Solution:

The integer parts must be equal: $8a + b = 12b + b$, so $8a = 12b$, that is $2a = 3b$. Since a and b are nonzero base-eight digits (at most 7), the only options are $(a, b) = (3, 2)$ and $(6, 4)$.

The fractional parts must also match: $\frac{c}{8} + \frac{d}{64} = \frac{b}{12} + \frac{a}{144}$, i.e. $\frac{8c+d}{64} = \frac{12b+a}{144}$. For $(a, b) = (6, 4)$ the right side is $\frac{54}{144} = \frac{24}{64}$, forcing $8c + d = 24$, which has no solution with both digits nonzero and less than 8. For $(a, b) = (3, 2)$ the right side is $\frac{27}{144} = \frac{12}{64}$, so $8c + d = 12$, giving $c = 1$ and $d = 4$.

Indeed $32.14_{\text{eight}} = 22.23_{\text{twelve}}$, and the requested number $\underline{a}\,\underline{b}\,\underline{c}$ is 321.

6. A circle is circumscribed around an isosceles triangle whose two congruent angles have degree measure x. Two points are chosen independently and uniformly at random on the circle, and a chord is drawn between them. The probability that the chord intersects the triangle is $\frac{14}{25}$. Find the difference between the largest and smallest possible values of x.

Solution:

Each inscribed angle of the triangle subtends an arc of twice its measure, so the vertices split the circle into arcs of $2x, 2x,$ and $360 - 4x$ degrees. The chord fails to intersect the triangle exactly when both random points fall in the same arc, which has probability

$$\left(\frac{2x}{360}\right)^2 + \left(\frac{2x}{360}\right)^2 + \left(\frac{360-4x}{360}\right)^2 = 1 - \frac{14}{25} = \frac{11}{25}.$$

Setting $y = \frac{x}{180}$, this reads $2y^2 + (1-2y)^2 = \frac{11}{25}$, which simplifies to $75y^2 - 50y + 7 = 0$, with roots $y = \frac{1}{5}$ and $y = \frac{7}{15}$. These give $x = 36$ and $x = 84$, both legitimate base angles of an isosceles triangle.

The requested difference is $84 - 36 = 48$.

7. For nonnegative integers a and b with $a + b \leq 6$, let $T(a, b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a, b)$, where a and b are nonnegative integers with $a + b \leq 6$. Find the remainder when S is divided by 1000.

Solution:

By the symmetry $\binom{6}{a+b} = \binom{6}{6-(a+b)}$, substituting $c = 6 - a - b$ turns the sum into

$$S = \sum_{a+b+c=6} \binom{6}{a}\binom{6}{b}\binom{6}{c}.$$

Each term counts the ways to choose a elements from one 6-element set, b from a second, and c from a third. Summed over all $a + b + c = 6$, this counts every way to choose 6 elements from the combined 18-element set, so $S = \binom{18}{6} = 18564.$

The remainder upon division by 1000 is $564.$

8. Two real numbers a and b are chosen independently and uniformly at random from the interval $(0, 75)$. Let O and P be two points in the plane with $OP = 200$. Let Q and R be points on the same side of line OP such that the degree measures of $\angle POQ$ and $\angle POR$ are a and b, respectively, and $\angle OQP$ and $\angle ORP$ are both right angles. The probability that $QR \leq 100$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Since $\angle OQP = \angle ORP = 90^\circ$, both Q and R lie on the circle with diameter $\overline{OP}$, whose radius is 100. The angle $\angle QOR = |a - b|$ is an inscribed angle in this circle, so the chord satisfies $QR = 2 \cdot 100 \cdot \sin |a - b|$. Because $|a - b| < 75^\circ$, the condition $QR \leq 100$, i.e. $\sin |a - b| \leq \frac{1}{2}$, is equivalent to $|a - b| \leq 30$.

In the 75×75 square of equally likely pairs (a, b), the region $|a - b| > 30$ consists of two right triangles with legs $75 - 30 = 45$, so the probability is

$$1 - \frac{45^2}{75^2} = 1 - \frac{9}{25} = \frac{16}{25}.$$

Therefore $m + n = 16 + 25 = 41$.

9. Let $a_{10} = 10$, and for each integer $n > 10$ let $a_n = 100a_{n-1} + n$. Find the least $n > 10$ such that a_n is a multiple of 99.

Solution:

Because $100 \equiv 1 \pmod{99}$, the recurrence gives $a_n \equiv a_{n-1} + n \pmod{99}$, so

$$\begin{aligned} a_n &\equiv 10 + 11 + \cdots + n \\ &= \frac{(n+10)(n-9)}{2} \pmod{99}. \end{aligned}$$

We need $99 \mid \frac{(n+10)(n-9)}{2}$. One of $n+10$ and $n-9$ is even, so this is the same as requiring 9 and 11 each to divide the product. Since the two factors differ by 19, they cannot both be multiples of 3.

So 9 must divide one factor entirely and 11 the other (or one factor is divisible by 99). Checking the cases: $99 \mid n-9$ first at $n = 108$; $99 \mid n+10$ first at $n = 89$; $9 \mid n+10$ with $11 \mid n-9$ first at $n = 53$; and $11 \mid n+10$ with $9 \mid n-9$ first at $n = 45$.

The least is $n = 45$, where $\frac{55 \cdot 36}{2} = 990$ is indeed a multiple of 99.

10. Let $z_1 = 18 + 83i$, $z_2 = 18 + 39i$, and $z_3 = 78 + 99i$, where $i = \sqrt{-1}$. Let z be the unique complex number with the properties that $\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z - z_2}{z - z_3}$ is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

Solution:

The argument of $\frac{z_3 - z_1}{z_2 - z_1}$ is the angle $\angle z_2 z_1 z_3$, and the argument of $\frac{z - z_2}{z - z_3}$ is the angle between $\overline{zz_2}$ and $\overline{zz_3}$. Their product is real exactly when these angles are equal or supplementary, which by the inscribed angle theorem happens exactly when z_1, z_2, z_3, and z are concyclic. So z lies on the circumcircle of z_1, z_2, z_3.

The segment from $18 + 39i$ to $18 + 83i$ is vertical, so its perpendicular bisector is the horizontal line $y = 61$. The segment from $z_2 = 18 + 39i$ to $z_3 = 78 + 99i$ has slope 1 and midpoint $(48, 69)$, so its perpendicular bisector is $y - 69 = -(x - 48)$. Setting $y = 61$ gives $x = 56$, so the center is $56 + 61i$.

The point of the circle with maximal imaginary part is directly above the center, so the real part of z is 56.

11. Consider arrangements of the 9 numbers $1, 2, 3, \dots, 9$ in a 3×3 array. For each such arrangement, let a_1, a_2, and a_3 be the medians of the numbers in rows $1, 2$, and 3, respectively, and then let m be the median of $\{a_1, a_2, a_3\}$. Let Q be the number of arrangements for which $m = 5$. Find the remainder when Q is divided by 1000.

Solution:

Rename each of $1, 2, 3, 4$ as L and each of $6, 7, 8, 9$ as G. If 5 is not a row median, then no row median equals 5, so $m \neq 5$. Thus 5's row must contain one L and one G (reading L5G in some order), and the other two rows must supply one median below 5 and one above. With the remaining three L's and three G's, those rows are either LLL and GGG, or LLG and LGG.

Count arrangements of letters: the three row types can be assigned to rows $1, 2, 3$ in $3! = 6$ ways, and the L5G row can be ordered in $3! = 6$ ways. In the first case LLL and GGG have 1 ordering each, giving $6 \cdot 6 \cdot 1 = 36$ patterns; in the second, LLG and LGG each have 3 orderings, giving $6 \cdot 6 \cdot 9 = 324$ patterns. That is 360 letter patterns in all.

Finally the four L's can be filled with $1, 2, 3, 4$ in $4!$ ways and the four G's with $6, 7, 8, 9$ in $4!$ ways, so $Q = 360 \cdot 24^2 = 207360$, whose remainder mod 1000 is 360.

12. Call a set S *product-free* if there do not exist $a, b, c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16, 20\}$ are product-free, whereas the sets $\{4, 16\}$ and $\{2, 8, 16\}$ are not product-free. Find the number of product-free subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Solution:

Since $1 \cdot 1 = 1$, no product-free set contains 1. Split by the least element t. If $t \geq 4$, any product of two elements is at least $16 > 10$, so every subset of $\{4, 5, \ldots, 10\}$ works: $2^7 = 128$ subsets, including the empty set.

If $t = 2:$ then $4 \notin S$ (as $2 \cdot 2 = 4$), while 7 and 8 are unrestricted (2 choices each). Among $\{3, 6, 9\}$, the constraints $2 \cdot 3 = 6$ and $3 \cdot 3 = 9$ leave exactly $\varnothing, \{3\}, \{6\}, \{9\}, \{6, 9\} - 5$ choices. Among $\{5, 10\}$, the constraint $2 \cdot 5 = 10$ leaves 3 choices. That gives $2 \cdot 2 \cdot 5 \cdot 3 = 60$ sets. If $t = 3:$ then $9 \notin S$ (as $3 \cdot 3 = 9$), and any subset of $\{4, 5, 6, 7, 8, 10\}$ may be added since all other products exceed $10:$ $2^6 = 64$ sets.

In total there are $128 + 60 + 64 = 252$ product-free subsets.

13. For every $m \geq 2$, let $Q(m)$ be the least positive integer with the following property: For every $n \geq Q(m)$, there is always a perfect cube k^3 in the range $n < k^3 \leq m \cdot n$. Find the remainder when

$$\sum_{m=2}^{2017} Q(m)$$

is divided by 1000.

Solution:

If $k^3 \leq n < (k+1)^3$, then the interval $(n, mn]$ contains the cube $(k+1)^3$ as long as $(k+1)^3 \leq mk^3$, i.e. $\left(1+\frac{1}{k}\right)^3 \leq m$. Since $\left(1+\frac{1}{k}\right)^3 \leq 8$ for all $k \geq 1$, every $m \geq 8$ has $Q(m) = 1$.

For $4 \leq m \leq 7 : n = 1$ fails since $(1, m]$ contains no cube, but for $n \geq 2$ the interval works: $8 \leq 4n$ covers $2 \leq n \leq 7$, and $\left(1+\frac{1}{k}\right)^3 \leq \frac{27}{8} < 4$ covers $k \geq 2$. So $Q(4) = Q(5) = Q(6) = Q(7) = 2$. For $m = 3 : n = 8$ fails (no cube in $(8, 24]$), while $27 \leq 3n$ covers $9 \leq n \leq 26$ and $\left(\frac{4}{3}\right)^3 < 3$ covers $k \geq 3$, so $Q(3) = 9$. For $m = 2 : n = 31$ fails (no cube in $(31, 62]$), while $64 \leq 2n$ covers $32 \leq n \leq 63$ and $\left(\frac{5}{4}\right)^3 < 2$ covers $k \geq 4$, so $Q(2) = 32$.

Therefore

$$\begin{aligned}\sum_{m=2}^{2017} Q(m) \\ &= 32 + 9 + 4 \cdot 2 + 2010 \cdot 1 \\ &= 2059,\end{aligned}$$

and the remainder is 59.

14. Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

Solution:

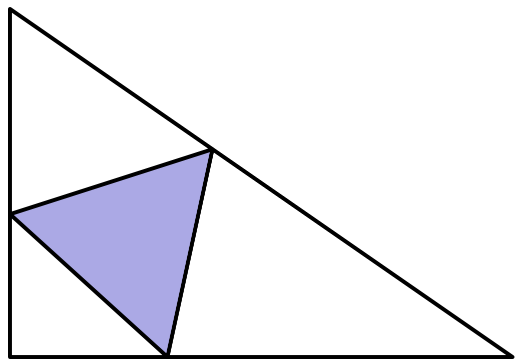
Exponentiating the first equation twice: $\log_a(\log_a 2) + \log_a 24 - 128 = a^{128}$ becomes $\log_a(24 \log_a 2) = 128 + a^{128}$, so $24 \log_a 2 = a^{128} \cdot a^{a^{128}}$, i.e. $2^{24} = a^{\left(a^{128} \cdot a^{a^{128}}\right)}$. Setting $t = a^{a^{128}}$, the right side is t^t and the left side is $\left(2^3\right)^{2^3}$, so by the strict monotonicity of t^t we get $a^{a^{128}} = 8$. Writing $c = \log_2 a > 0$, this says $c \cdot 2^{128c} = 3$, which is increasing in c and satisfied by $c = \frac{3}{64}$: indeed $\frac{3}{64} \cdot 2^6 = 3$. So $a = 2^{\frac{3}{64}}$.

The second equation gives $x = a^{a^{256}}$. Here $a^{256} = 2^{\frac{256 \cdot 3}{64}} = 2^{12} = 4096$, so

$$x = a^{4096} = 2^{\frac{4096 \cdot 3}{64}} = 2^{192}.$$

Clearly $2^{192} \equiv 0 \pmod 8$. By Euler's theorem $2^{100} \equiv 1 \pmod{125}$, so $2^{192} \equiv 2^{-8} \pmod{125}$, the inverse of $256 \equiv 6$. Since $6 \cdot 21 = 126 \equiv 1 \pmod{125}$, we get $2^{192} \equiv 21 \pmod{125}$. The unique residue mod 1000 that is 0 mod 8 and 21 mod 125 is 896.

15. The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}, 5$, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.



Solution:

Place the right angle at the origin with vertices $(0,0)$, $(5,0)$, and $(0,2\sqrt{3})$, so the hypotenuse lies on the line $2\sqrt{3}\,x+5y=10\sqrt{3}$. Let the equilateral triangle's side between the two legs have endpoints $(s\cos\theta, 0)$ and $(0, s\sin\theta)$, where s is the side length. Its midpoint is $\frac{s}{2}(\cos\theta, \sin\theta)$, and moving a distance $\frac{\sqrt{3}}{2}s$ perpendicular to the side places the third vertex at $\frac{s}{2}\cdot\left(\cos\theta+\sqrt{3}\sin\theta,\ \sin\theta+\sqrt{3}\cos\theta\right)$.

Substituting this vertex into the hypotenuse equation and simplifying gives

$$s=\frac{20\sqrt{3}}{7\sqrt{3}\cos\theta+11\sin\theta}.$$

The denominator is at most $\sqrt{(7\sqrt{3})^2+11^2}=\sqrt{268}=2\sqrt{67}$, attained for an admissible θ, so the minimum side length satisfies $s^2=\frac{(10\sqrt{3})^2}{67}=\frac{300}{67}$.

The minimum area is $\frac{\sqrt{3}}{4}\cdot\frac{300}{67}=\frac{75\sqrt{3}}{67}$, so $m+n+p=75+67+3=145.$

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