

2018 AIME II

Time limit: 180 minutes

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1. Points A , B , and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C , Paul starting at B and running toward C , and Eve starting at C and running toward A . When Paul meets Eve, he turns around and runs toward A . Paul and Ina both arrive at B at the same time. Find the number of meters from A to B .
2. Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.
3. Find the sum of all positive integers $b < 1000$ such that the base- b integer 36_b is a perfect square and the base- b integer 27_b is a perfect cube.

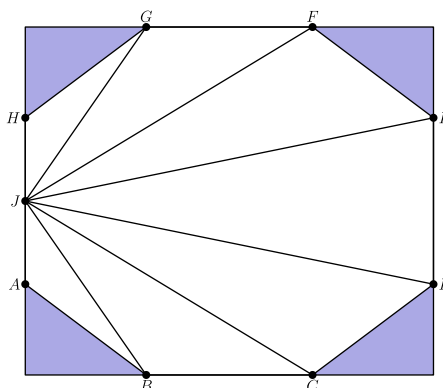
4. In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. Then $K = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.
5. Suppose that x , y , and z are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96 + 24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x + y + z = a + bi$. Find $a^2 + b^2$.
6. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a - 2)x^2 + (-4a + 3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. Triangle ABC has side lengths $AB = 9$, $BC = 5\sqrt{3}$, and $AC = 12$. Points $A = P_0, P_1, P_2, \dots, P_{2450} = B$ are on segment $\overline{AB}$ with P_k between P_{k-1} and P_{k+1} for $k = 1, 2, \dots, 2449$, and points $A = Q_0, Q_1, Q_2, \dots, Q_{2450} = C$ are on segment $\overline{AC}$ with Q_k between Q_{k-1} and Q_{k+1} for $k = 1, 2, \dots, 2449$. Furthermore, each segment $\overline{P_k Q_k}, k = 1, 2, \dots, 2449$, is parallel to $\overline{BC}$. The segments cut the triangle into 2450 regions, consisting of 2449 trapezoids and 1 triangle. Each of the 2450 regions has the same area. Find the number of segments $\overline{P_k Q_k}, k = 1, 2, \dots, 2450$, that have rational length.
8. A frog is positioned at the origin in the coordinate plane. From the point (x, y) , the frog can jump to any of the points $(x + 1, y), (x + 2, y), (x, y + 1)$, or $(x, y + 2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0, 0)$ and ends at $(4, 4)$.

9. Octagon $ABCDEFGH$ with side lengths $AB = CD = EF = GH = 10$ and $BC = DE = FG = HA = 11$ is formed by removing four 6-8-10 triangles from the corners of a 23×27 rectangle with side $\overline{AH}$ on a short side of the rectangle, as shown. Let J be the midpoint of $\overline{AH}$, and partition the octagon into 7 triangles by drawing segments $\overline{JB}$, $\overline{JC}$, $\overline{JD}$, $\overline{JE}$, $\overline{JF}$, and $\overline{JG}$. Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.



10. Find the number of functions $f(x)$ from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$ that satisfy $f(f(x)) = f(f(f(x)))$ for all x in $\{1, 2, 3, 4, 5\}$.
11. Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k .

12. Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P , and that the sum of the areas of triangles APB and CPD equals the sum of the areas of triangles BPC and APD . Find the area of quadrilateral $ABCD$.
13. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
14. The incircle ω of triangle ABC is tangent to $\overline{BC}$ at X . Let $Y \neq X$ be the other intersection of $\overline{AX}$ with ω . Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y . Assume that $AP = 3$, $PB = 4$, $AC = 8$, and $AQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. Find the number of functions f from $\{0, 1, 2, 3, 4, 5, 6\}$ to the integers such that $f(0) = 0$, $f(6) = 12$, and

$$\begin{aligned} |x - y| &\leq |f(x) - f(y)| \\ &\leq 3|x - y| \end{aligned}$$

for all x and y in $\{0, 1, 2, 3, 4, 5, 6\}$.

Solutions: <https://live.poshenloh.com/past-contests/aime/2018II/solutions>

