2018 AIME II Solutions

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1. Points $A, B,$ and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C, Paul starting at B and running toward C, and Eve starting at C and running toward A. When Paul meets Eve, he turns around and runs toward A. Paul and Ina both arrive at B at the same time. Find the number of meters from A to B.

Solution:

Let $x = AB$, so $BC = 1800 - x$, and let Eve's speed be v, so Ina runs at $2v$ and Paul at $4v$. Paul and Eve start at B and C running toward each other, so together they cover the $1800 - x$ meters between them, with Paul covering $\frac{4}{5}$ of it. Paul then retraces that distance back to B, so when he reaches B he has run $\frac{8}{5}(1800 - x)$ meters in total.

Ina reaches B at the same moment, having run x meters. Since Paul runs twice as fast as Ina, he has run $2x$ meters in that time. Therefore

$$\frac{8}{5}(1800 - x) = 2x,$$

which gives $8 \cdot 1800 = 18x$, so $x = 800$.

2. Let $a_0 = 2, a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

Solution:

Computing successive terms gives

$$2,\ 5,\ 8,\ 5,\ 6,\ 10,\ 7,\ 4,\ 7,\ 6,$$
$$2,\ 5,\ 8,\ \ldots$$

Since $(a_{10}, a_{11}, a_{12}) = (2, 5, 8) = (a_0, a_1, a_2)$ and each term depends only on the previous three, the sequence is periodic with period 10.

Therefore $a_{2018} = a_8 = 7, a_{2020} = a_0 = 2$, and $a_{2022} = a_2 = 8$, so the product is $7 \cdot 2 \cdot 8 = 112$.

3. Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

Solution:

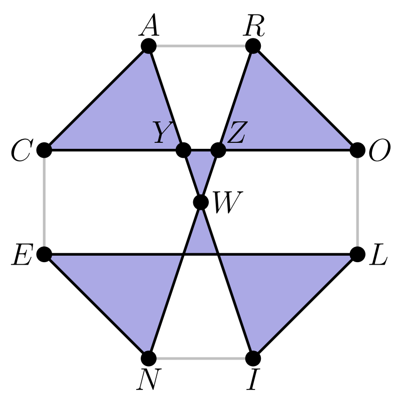
The conditions say $3b + 6$ is a perfect square and $2b + 7$ is a perfect cube. Since $2b + 7$ is odd and $b < 1000$ forces $2b + 7 < 2007$, the cube must be one of $1, 27, 125, 343, 729, 1331$, giving $b = -3, 10, 59, 168, 361, 662$.

The corresponding values of $3b + 6$ for the positive candidates are $36, 183, 510, 1089, 1992$, and only $36 = 6^2$ (for $b = 10$) and $1089 = 33^2$ (for $b = 361$) are perfect squares. The requested sum is $10 + 361 = 371$.

4. In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. Then $K = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Solution:

Since the interior angles are all 135° and the $\sqrt{2}$ sides are diagonals of unit squares, the octagon fits on a lattice: $C = (0,0), A = (1,1), R = (2,1), O = (3,0), L = (3,-1), I = (2,-2), N = (1,-2), E = (0,-1)$. The path $CORNELIA$ is carried to itself by the 180° rotation about $(\frac{3}{2}, -\frac{1}{2})$. Let Y and Z be the points where AI and RN cross CO, and let $W = AI \cap RN$. Segment AI has slope -3, so $Y = (\frac{4}{3}, 0)$, and by symmetry $Z = (\frac{5}{3}, 0)$ and $W = (\frac{3}{2}, -\frac{1}{2})$.



The six enclosed regions are the four congruent corner triangles like CAY and the two small congruent triangles like YZW. Triangle CAY has base $CY = \frac{4}{3}$ and height 1, so its area is $\frac{2}{3}$. Triangle YZW has base $YZ = \frac{1}{3}$ and height $\frac{1}{2}$, so its area is $\frac{1}{12}$. Therefore

$$\begin{aligned} K &= 4 \cdot \frac{2}{3} + 2 \cdot \frac{1}{12} \\ &= \frac{8}{3} + \frac{1}{6} = \frac{17}{6}, \end{aligned}$$

and $a + b = 17 + 6 = 23$.

5. Suppose that $x, y,$ and z are complex numbers such that $xy = -80 - 320i, yz = 60,$ and $zx = -96 + 24i,$ where $i = \sqrt{-1}.$ Then there are real numbers a and b such that $x + y + z = a + bi.$ Find $a^2 + b^2.$

Solution:

Multiplying the three equations gives

$$\begin{aligned}(xyz)^2 &= (-80 - 320i)(60) \\ &\quad \cdot (-96 + 24i) \\ &= 80 \cdot 60 \cdot 24 \\ &\quad \cdot \ (-1 - 4i)(-4 + i) \\ &= 240^2(16 + 30i).\end{aligned}$$

Since $16 + 30i = (5 + 3i)^2$, we get $xyz = \pm 240(5 + 3i).$

Dividing xyz by each given product yields

$$\begin{aligned}x &= \frac{xyz}{yz} = \pm(20 + 12i), \\ y &= \frac{xyz}{zx} = \pm(-10 - 10i), \\ z &= \frac{xyz}{xy} = \pm(-3 + 3i),\end{aligned}$$

with matching signs. Hence $x + y + z = \pm(7 + 5i)$, so $(a, b) = \pm(7, 5)$ and $a^2 + b^2 = 49 + 25 = 74.$

6. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a-2)x^2 \\ + (-4a+3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Group the terms by whether they involve a :

$$\begin{aligned} &(x^4 - 2x^2 + 3x - 2) \\ &\quad + 2a(x^3 + x^2 - 2x) \\ &= (x-1)(x+2)(x^2 - x + 1) \\ &\quad + 2ax(x-1)(x+2), \end{aligned}$$

so the polynomial factors as $(x-1)(x+2) \cdot \left(x^2 + (2a-1)x + 1\right)$.

All four roots are real exactly when the quadratic factor has real roots, i.e. when $(2a-1)^2 - 4 \geq 0$, which means $a \leq -\frac{1}{2}$ or $a \geq \frac{3}{2}$. The excluded interval $\left(-\frac{1}{2}, \frac{3}{2}\right)$ has length 2 inside $[-20, 18]$, which has length 38, so the probability is $\frac{36}{38} = \frac{18}{19}$. The requested sum is $18 + 19 = 37$.

7. Triangle ABC has side lengths $AB = 9$, $BC = 5\sqrt{3}$, and $AC = 12$. Points $A = P_0$, $P_1, P_2, \ldots, P_{2450} = B$ are on segment $\overline{AB}$ with P_k between P_{k-1} and P_{k+1} for $k = 1, 2, \ldots, 2449$, and points $A = Q_0, Q_1, Q_2, \ldots, Q_{2450} = C$ are on segment $\overline{AC}$ with Q_k between Q_{k-1} and Q_{k+1} for $k = 1, 2, \ldots, 2449$. Furthermore, each segment $\overline{P_kQ_k}$, $k = 1, 2, \ldots, 2449$, is parallel to $\overline{BC}$. The segments cut the triangle into 2450 regions, consisting of 2449 trapezoids and 1 triangle. Each of the 2450 regions has the same area. Find the number of segments $\overline{P_kQ_k}$, $k = 1, 2, \ldots, 2450$, that have rational length.

Solution:

Since the 2450 regions have equal areas, triangle AP_kQ_k (the union of the first k regions) has area $\frac{k}{2450}$ of triangle ABC. Each triangle AP_kQ_k is similar to ABC, and lengths scale as the square root of areas, so

$$\begin{aligned} P_kQ_k &= 5\sqrt{3}\sqrt{\frac{k}{2450}} \\ &= 5\sqrt{3} \cdot \frac{\sqrt{k}}{35\sqrt{2}} \\ &= \frac{\sqrt{6k}}{14}. \end{aligned}$$

This is rational exactly when $6k$ is a perfect square, which happens exactly when $k = 6j^2$ for a positive integer j. The condition $6j^2 \leq 2450$ gives $j^2 \leq 408$, so $j = 1, 2, \ldots, 20$. There are 20 such segments.

8. A frog is positioned at the origin in the coordinate plane. From the point (x, y), the frog can jump to any of the points $(x+1, y)$, $(x+2, y)$, $(x, y+1)$, or $(x, y+2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0,0)$ and ends at $(4,4)$.

Solution:

The horizontal jumps are steps of 1 or 2 summing to 4, so as a multiset they are $\{1,1,1,1\}, \{1,1,2\}$, or $\{2,2\}$, and the same holds for the vertical jumps. For any choice of the two multisets, every ordering of all the jumps is a valid sequence, and the number of orderings is the multinomial coefficient of the combined multiset.

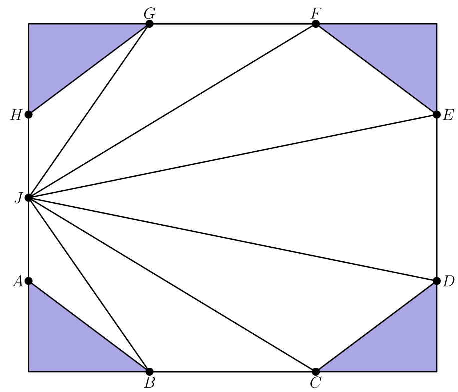
The nine cases give

$$\binom{8}{4} = 70,$$
$$\frac{7!}{4!\,2!} = 105 \text{ (twice)},$$
$$\frac{6!}{4!\,2!} = 15 \text{ (twice)},$$

$$\frac{6!}{2!\,2!} = 180,$$
$$\frac{5!}{2!\,2!} = 30 \text{ (twice)},$$
$$\binom{4}{2} = 6.$$

The total is $70 + 2 \cdot 105 + 2 \cdot 15 + 180 + 2 \cdot 30 + 6 = 556.$

9. Octagon $ABCDEFGH$ with side lengths $AB = CD = EF = GH = 10$ and $BC = DE = FG = HA = 11$ is formed by removing four 6-8-10 triangles from the corners of a 23×27 rectangle with side $\overline{AH}$ on a short side of the rectangle, as shown. Let J be the midpoint of $\overline{AH}$, and partition the octagon into 7 triangles by drawing segments $\overline{JB}, \overline{JC}, \overline{JD}, \overline{JE}, \overline{JF}$, and $\overline{JG}$. Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.



Solution:

Each of the 7 triangles has J as a vertex, and the centroid of a triangle JVW lies on the segment from J to the midpoint of $\overline{VW}$, two-thirds of the way out. So the centroid heptagon is the image of the heptagon S formed by the midpoints of $\overline{AB}, \overline{BC}, \ldots, \overline{GH}$ under a dilation centered at J with ratio $\frac{2}{3}$, and its area is $\frac{4}{9}[S]$.

Place the rectangle with $A = (0,6), B = (8,0), C = (19,0), D = (27,6), E = (27,17), F = (19,23), G = (8,23), H = (0,17)$, so $J = (0, \frac{23}{2})$. The midpoints are $(4,3), (\frac{27}{2}, 0), (23,3), (27, \frac{23}{2}), (23,20), (\frac{27}{2}, 23), (4,20)$. The vertical segments at $x = 4, x = \frac{27}{2}$, and $x = 23$ have lengths $17, 23$, and 17, cutting S into two trapezoids of height $\frac{19}{2}$ and a triangle of height 4 :

$$[S] = 2 \cdot \frac{17+23}{2} \cdot \frac{19}{2} + \frac{17 \cdot 4}{2} = 380 + 34 = 414.$$

The requested area is $\frac{4}{9} \cdot 414 = 184.$

10. Find the number of functions $f(x)$ from $\{1,2,3,4,5\}$ to $\{1,2,3,4,5\}$ that satisfy $f(f(x)) = f(f(f(x)))$ for all x in $\{1,2,3,4,5\}$.

Solution:

Applying f to $f(f(x)) = f(f(f(x)))$ repeatedly shows the condition means that $f(f(x))$ is a fixed point of f for every x. So the elements organize into levels: a nonempty set of i fixed points, then j elements whose image is a fixed point (but which are not fixed), and the remaining $5-i-j$ elements, each of which must map to one of the j middle elements.

For given i and j there are $\binom{5}{i}$ choices of fixed points, $\binom{5-i}{j}$ choices of the middle level, i^j maps from the middle level to the fixed points, and $j^{\,5-i-j}$ maps for the rest. Summing $\binom{5}{i}\binom{5-i}{j}\, i^j\, j^{\,5-i-j}$ over the valid pairs (all $i \geq 1, j \geq 1,$ plus the identity case $i=5$) gives

$$\begin{aligned} &20 + 120 + 60 + 5 \\ &\quad + 60 + 240 + 80 \\ &\quad + 60 + 90 + 20 + 1 = 756. \end{aligned}$$

11. Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \le k \le 5$, at least one of the first k terms of the permutation is greater than k.

Solution:

The condition fails exactly when the first k terms are a permutation of $\{1, \ldots, k\}$ for some $k \le 5$. For a permutation of $1, \ldots, n$, let k be the smallest length for which the prefix is $\{1, \ldots, k\}$ (the full length n always works), and let c_n be the number of permutations whose smallest such k is n. We want c_6.

Every permutation of $1, \ldots, n$ decomposes uniquely as a minimal prefix of length k (c_k choices) followed by any arrangement of the remaining $n-k$ values, so

$$\sum_{k=1}^{n} c_k \,(n-k)! = n!.$$

Starting from $c_1 = 1$, this gives $c_2 = 1, c_3 = 3, c_4 = 13, c_5 = 71$, and

$$\begin{aligned} c_6 &= 720 \\ &\quad - (120 \cdot 1 + 24 \cdot 1 + 6 \cdot 3 + 2 \cdot 13 + 1 \cdot 71) \\ &= 720 - 259 = 461. \end{aligned}$$

12. Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of triangles APB and CPD equals the sum of the areas of triangles BPC and APD. Find the area of quadrilateral $ABCD$.

Solution:

Let $a = AP, b = BP, c = CP, d = DP$, and let $\theta = \angle CPD$. Since $\sin(\pi - \theta) = \sin\theta$, the equal-area condition $\frac{1}{2}(ab + cd)\sin\theta = \frac{1}{2}(ad + bc)\sin\theta$ simplifies to $(a - c)(d - b) = 0$. By symmetry assume $a = c$.

The law of cosines in triangles BPC and APB (whose angles at P are supplementary) gives $a^2 + b^2 + 2ab\cos\theta = 196$ and $a^2 + b^2 - 2ab\cos\theta = 100$, so $a^2 + b^2 = 148$ and $ab\cos\theta = 24$. Similarly triangles APD and CPD give $a^2 + d^2 = 180$ and $ad\cos\theta = 40$. Dividing, $\frac{d}{b} = \frac{5}{3}$, while subtracting gives $d^2 - b^2 = 32$; hence $b = 3\sqrt{2}, d = 5\sqrt{2}, a^2 = 130$, and $\cos^2\theta = \frac{24^2}{130\cdot 18} = \frac{16}{65}$, so $\sin\theta = \frac{7}{\sqrt{65}}$.

The total area is

$$\begin{aligned} \frac{1}{2}(a+c)(b+d) \\ \cdot \sin\theta \\ &= a(b+d)\sin\theta \\ &= \sqrt{130}\cdot 8\sqrt{2} \\ &\cdot \frac{7}{\sqrt{65}} \\ &= 112. \end{aligned}$$

13. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Let a be the probability that the total number of rolls is odd; let b be that probability given that the first roll is a 1, and c given that the first two rolls are 1-2 (in each case counting all rolls). Condition on the next roll, noting that whenever the count restarts, the rolls already used flip the required parity. Starting fresh: a 1 leads to state b; anything else uses one roll, after which an even continuation is needed. After a 1 : another 1 means the first roll is wasted, needing an even continuation of the b-type; a 2 leads to c; anything else wastes both rolls. After 1-2 : a 3 finishes in 3 rolls (odd); a 1 restarts at the b-state with two wasted rolls; anything else wastes all three. Thus

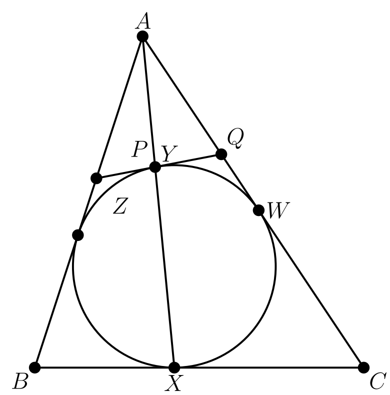
$$\begin{aligned} a &= \frac{1}{6}b + \frac{5}{6}(1-a), \\ b &= \frac{1}{6}(1-b) + \frac{1}{6}c + \frac{4}{6}a, \\ c &= \frac{1}{6}b + \frac{1}{6} + \frac{4}{6}(1-a). \end{aligned}$$

The first equation gives $b = 11a - 5$; substituting the third into the second yields $41b = 11 + 20a$, so $41(11a-5) = 11 + 20a$, giving $431a = 216$ and $a = \frac{216}{431}$. Since 431 is prime, $m + n = 216 + 431 = 647$.

14. The incircle ω of triangle ABC is tangent to $\overline{BC}$ at X. Let $Y \neq X$ be the other intersection of $\overline{AX}$ with ω. Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y. Assume that $AP = 3$, $PB = 4$, $AC = 8$, and $AQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Let ω touch $\overline{AB}$ at Z and $\overline{AC}$ at W, and set $\alpha = \angle BAX$ and $\beta = \angle AXC$. The tangent-chord angle between PQ and chord XY equals the one between BC and XY, so $\angle QYX = \angle YXC = \beta$, and vertical angles give $\angle AYP = \beta$. In triangle APY the law of sines gives $PY = AP \frac{\sin\alpha}{\sin\beta}$, and by equal tangents $PZ = PY$, so $\frac{AZ}{AP} = 1 + \frac{PY}{AP} = 1 + \frac{\sin\alpha}{\sin\beta}$. In triangle ABX, since $\angle AXB = 180^\circ - \beta$, similarly $BX = AB \frac{\sin\alpha}{\sin\beta}$, and $BZ = BX$ gives $\frac{AZ}{AB} = 1 - \frac{\sin\alpha}{\sin\beta}$.



Adding the two relations, $\frac{AZ}{AP} + \frac{AZ}{AB} = 2$, so with $AP = 3$ and $AB = 7$ we get $AZ\left(\frac{1}{3} + \frac{1}{7}\right) = 2$, hence $AZ = \frac{21}{5}$. The identical argument on side AC (using $\angle XAC$ in triangles AQY and ACX) gives $\frac{AW}{AQ} + \frac{AW}{AC} = 2$, and $AW = AZ = \frac{21}{5}$ by equal tangents from A. Therefore

$$\frac{1}{AQ} = \frac{10}{21} - \frac{1}{8} = \frac{59}{168},$$

so $AQ = \frac{168}{59}$ and $m + n = 168 + 59 = 227$.

15. Find the number of functions f from $\{0,1,2,3,4,5,6\}$ to the integers such that $f(0)=0, f(6)=12$, and

$$|x-y| \leq |f(x)-f(y)| \\ \leq 3|x-y|$$

for all x and y in $\{0,1,2,3,4,5,6\}$.

Solution:

Let $d_i = f(i)-f(i-1)$, so each $|d_i| \in \{1,2,3\}$ and $d_1+\cdots+d_6=12$. If k of the differences were negative, the sum would be at most $3(6-k)-k=18-4k$, so $k \leq 1$. If no difference is negative, the solutions of $a+b+c=6, a+2b+3c=12$ (with a,b,c counting 1s, 2s, 3s) are $(0,6,0),(1,4,1),(2,2,2),(3,0,3)$, and all such orderings satisfy every pair condition, giving

$$\binom{6}{0,6,0}+\binom{6}{1,4,1} \\ +\binom{6}{2,2,2}+\binom{6}{3,0,3} \\ =1+30+90+20=141.$$

If $d_1<0$, then $|f(2)| \geq 2$ with $f(2) \leq d_1+3$ forces $d_1=-1$ and $d_2=3$; the remaining four differences are positive and sum to 10, giving $\binom{4}{1,0,3}+\binom{4}{0,2,2}=4+6=10$ functions, and the case $d_6<0$ is symmetric: 20 in all. Finally, if $d_{n+1}<0$ for some $n=1,2,3,4$, the pair conditions $|f(n+1)-f(n-1)| \geq 2$ and $|f(n+2)-f(n)| \geq 2$ force $d_{n+1}=-1$ and $d_n=d_{n+2}=3$. The other three differences are positive and sum to 7, achievable as two 3s and a 1 or a 3 and two 2s, each in 3 orders: 6 ways for each of the 4 positions, or 24 functions.

The total is $141+20+24=185$.

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