

2019 AIME I

Time limit: 180 minutes

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1. Consider the integer

$$N = 9 + 99 + 999 + 9999 \\ + \cdots + \underbrace{99 \dots 99}_{321 \text{ digits}}.$$

Find the sum of the digits of N .

2. Jenn randomly chooses a number J from $1, 2, 3, \dots, 19, 20$. Bela then randomly chooses a number B from $1, 2, 3, \dots, 19, 20$ distinct from J . The value of $B - J$ is at least 2 with a probability that can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.
4. A soccer team has 22 available players. A fixed set of 11 players starts the game, while the other 11 are available as substitutes. During the game, the coach may make as many as 3 substitutions, where any one of the 11 players in the game is replaced by one of the substitutes. No player removed from the game may reenter the game, although a substitute entering the game may be replaced later. No two substitutions can happen at the same time. The players involved and the order of the substitutions matter. Let n be the number of ways the coach can make substitutions during the game (including the possibility of making no substitutions). Find the remainder when n is divided by 1000.
5. A moving particle starts at the point $(4, 4)$ and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a, b) , it moves at random to one of the points $(a - 1, b)$, $(a, b - 1)$, or $(a - 1, b - 1)$, each with probability $\frac{1}{3}$, independently of its previous moves. The probability that it will hit the coordinate axes at $(0, 0)$ is $\frac{m}{3^n}$, where m and n are positive integers, and m is not divisible by 3. Find $m + n$.

6. In convex quadrilateral $KLMN$, side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO .

7. There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2 \log_{10}(\gcd(x, y)) = 60$$

$$\begin{aligned} \log_{10} y \\ + 2 \log_{10}(\text{lcm}(x, y)) = 570. \end{aligned}$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x , and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y . Find $3m + 2n$.

8. Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
9. Let $\tau(n)$ denote the number of positive integer divisors of n . Find the sum of the six least positive integers n that are solutions to $\tau(n) + \tau(n + 1) = 7$.

10. For distinct complex numbers $z_1, z_2, \dots, z_{673}$, the polynomial

$$(x - z_1)^3(x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \leq j < k \leq 673} z_j z_k \right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. In $\triangle ABC$, the sides have integer lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An excircle of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω , and the other two excircles are both externally tangent to ω . Find the minimum possible value of the perimeter of $\triangle ABC$.

12. Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z , $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.
13. Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c , and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.
14. Find the least odd prime factor of $2019^8 + 1$.
15. Let $\overline{AB}$ be a chord of a circle ω , and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω . Circle ω_2 passes through B and P and is internally tangent to ω . Circles ω_1 and ω_2 intersect at points P and Q . Line PQ intersects ω at X and Y . Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solutions: <https://live.poshenloh.com/past-contests/aime/2019I/solutions>

