2019 AIME II Solutions

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1. Two different points, C and D, lie on the same side of line AB so that $\triangle ABC$ and $\triangle BAD$ are congruent with $AB = 9, BC = AD = 10$, and $CA = DB = 17$. The intersection of these two triangular regions has area $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Place $A = (0, 0)$ and $B = (9, 0)$. From $CA = 17$ and $BC = 10$, solving $x^2 + y^2 = 289$ and $(x - 9)^2 + y^2 = 100$ gives $C = (15, 8)$. The congruence $\triangle ABC \cong \triangle BAD$ swaps A and B, so D is the reflection of C across the line $x = \frac{9}{2}$, namely $D = (-6, 8)$.

A point lies in triangle ABC exactly when it is on or above $\overline{AB}$, on B's side of line AC, and on A's side of line BC; similarly for triangle BAD. In the overlap the binding constraints are $y \geq 0$, line AC, and line BD, so the intersection is the triangle with base $\overline{AB}$ and apex $E = AC \cap BD$. Line AC is $y = \frac{8x}{15}$ and line BD is $y = -\frac{8(x-9)}{15}$, which meet at $E = \left(\frac{9}{2}, \frac{12}{5}\right)$.

The area is

$$\frac{1}{2} \cdot 9 \cdot \frac{12}{5} = \frac{54}{5},$$

so $m + n = 54 + 5 = 59$.

2. Lily pads $1, 2, 3, \ldots$ lie in a row on a pond. A frog makes a sequence of jumps starting on pad 1. From any pad k the frog jumps to either pad $k+1$ or pad $k+2$ chosen randomly with probability $\frac{1}{2}$ and independently of other jumps. The probability that the frog visits pad 7 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

Let p_k be the probability that the frog visits pad k. The frog lands on pad k in exactly one of two disjoint ways: it visits pad $k-1$ and jumps $+1$ from there (if it jumps $+2$, pad k is skipped forever), or it skips pad $k-1$ entirely, which requires visiting pad $k-2$ and jumping $+2$ from it, landing on pad k. Hence

$$p_k = \tfrac{1}{2}p_{k-1} + \tfrac{1}{2}p_{k-2},$$
$$p_1 = 1, \quad p_2 = \tfrac{1}{2}.$$

Iterating: $p_3 = \frac{3}{4}$, $p_4 = \frac{5}{8}$, $p_5 = \frac{11}{16}$, $p_6 = \frac{21}{32}$, and $p_7 = \frac{43}{64}$. Since $\gcd(43, 64) = 1$, the answer is $43 + 64 = 107$.

3. Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following system of equations:

$$abc = 70,$$

$$cde = 71,$$

$$efg = 72.$$

Solution:

Since 71 is prime, in $cde = 71$ one of the three factors is 71 and the other two equal 1. But c divides $abc = 70$ and e divides $efg = 72$, and 71 divides neither 70 nor 72. So $c = e = 1$ and $d = 71$.

The system reduces to $ab = 70$ and $fg = 72$. Each divisor a of 70 determines b, giving $\tau(70) = 8$ ordered pairs, and likewise $\tau(72) = 12$ ordered pairs (f, g). The total is $8 \cdot 12 = 96$.

4. A standard six-sided fair die is rolled four times. The probability that the product of all four numbers rolled is a perfect square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

The product is a perfect square exactly when each of the primes $2, 3,$ and 5 appears with even exponent. Only a roll of 5 contributes the prime $5,$ so the number of 5s is even: $0, 2,$ or $4.$ Classify the other values by the parities of their exponents of 2 and $3:$ rolls of 1 and 4 contribute $(0,0),$ a 2 contributes $(1,0),$ a 3 contributes $(0,1),$ and a 6 contributes $(1,1).$ The exponent of 2 is even iff the count of 2s plus the count of 6s is even, and similarly for 3s and 6s, so a collection of non-5 rolls works exactly when the counts of 2s, 3s, and 6s are all even or all odd.

With no 5s, all four rolls come from $\{1,2,3,4,6\}.$ All-even cases: no 2s, 3s, or 6s gives $2^4=16$ sequences (each roll is 1 or 4); exactly two of a single kind gives $3\cdot\frac{4!}{2!\,2!}\cdot 2^2=72;$ two of each of two kinds gives $3\cdot\frac{4!}{2!\,2!}=18;$ four of one kind gives $3.$ All-odd case: one $2,$ one $3,$ one $6,$ and one roll from $\{1,4\}$ gives $4!\cdot 2=48.$ Subtotal $16+72+18+3+48=157.$ With two 5s, choose their positions in $\binom{4}{2}=6$ ways; the other two rolls must lie in the same parity class, giving $2^2+1+1+1=7$ ordered pairs, for 42 sequences. With four 5s there is 1 sequence.

In total $157+42+1=200$ of the $6^4=1296$ sequences work, so the probability is $\frac{200}{1296}=\frac{25}{162}$, and $m+n=25+162=187.$

5. Four ambassadors and one advisor for each of them are to be seated at a round table with 12 chairs numbered in order 1 to $12.$ Each ambassador must sit in an even-numbered chair. Each advisor must sit in a chair adjacent to his or her ambassador. There are N ways for the 8 people to be seated at the table under these conditions. Find the remainder when N is divided by $1000.$

Solution:

The six even chairs form a cycle (chair 12 is adjacent to chair 1), and each odd chair lies between two consecutive even chairs. The ambassadors occupy 4 of the 6 even chairs, and each advisor must take one of the two odd chairs flanking their ambassador, with all choices distinct. For a maximal block of k consecutive occupied even chairs, the k occupants choose among the $k+1$ odd chairs touching the block; recording each choice as left or right, a conflict occurs exactly when someone picks right and their neighbor picks left, so the valid patterns are the strings of Ls followed by Rs: $k+1$ of them.

Now case on the two empty even chairs among the six positions. If they are adjacent (6 ways), the occupied chairs form one block of $4,$ giving 5 patterns. If they are separated by one chair (6 ways), the blocks have sizes 1 and $3,$ giving $2 \cdot 4 = 8$ patterns. If they are opposite (3 ways), the blocks have sizes 2 and $2,$ giving $3 \cdot 3 = 9$ patterns. The number of seat configurations is $6 \cdot 5 + 6 \cdot 8 + 3 \cdot 9 = 105.$

Finally, the four ambassador-advisor pairs can be assigned to the four chosen even chairs in $4! = 24$ ways, so $N = 105 \cdot 24 = 2520,$ and the remainder modulo 1000 is $520.$

6. In a Martian civilization, all logarithms whose bases are not specified are assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56$$

$$\log_{\log x}(x) = 54$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.

Solution:

Let $y = \log_b x$. By the change-of-base formula,

$$\begin{aligned}\log_{\log x}(x) &= \frac{\log_b x}{\log_b(\log_b x)} \\ &= \frac{y}{\log_b y} = 54,\end{aligned}$$

so $\log_b y = \frac{y}{54}$. The first equation says $3\left(\frac{1}{2}\log_b x + \log_b(\log_b x)\right) = 56$, that is, $\frac{y}{2} + \log_b y = \frac{56}{3}$.

Substituting $\log_b y = \frac{y}{54}$ gives $\frac{y}{2} + \frac{y}{54} = \frac{56}{3}$, so $\frac{28y}{54} = \frac{56}{3}$ and $y = 36$. Then $\log_b 36 = \frac{36}{54} = \frac{2}{3}$, so $b^{\frac{2}{3}} = 36$ and $b = 36^{\frac{3}{2}} = 216$.

7. Triangle ABC has side lengths $AB = 120, BC = 220$, and $AC = 180$. Lines ℓ_A, ℓ_B, and ℓ_C are drawn parallel to $\overline{BC}, \overline{AC}$, and $\overline{AB}$, respectively, such that the intersections of ℓ_A, ℓ_B, and ℓ_C with the interior of $\triangle ABC$ are segments of lengths 55, 45, and 15, respectively. Find the perimeter of the triangle whose sides lie on lines ℓ_A, ℓ_B, and ℓ_C.

Solution:

For a point P, let α be the distance from P to line BC divided by the length of the altitude from A, and define β (to CA) and γ (to AB) similarly; then $\alpha + \beta + \gamma = 1$ for points inside, since $[PBC] + [PCA] + [PAB] = [ABC]$. A chord parallel to $\overline{BC}$ at level α cuts off a triangle at A similar to ABC with ratio $1 - \alpha$, so its length is $220(1 - \alpha)$. The chord of length 55 gives $1 - \alpha = \frac{1}{4}$, so ℓ_A is the line $\alpha = \frac{3}{4}$; similarly $45 = 180(1 - \beta)$ puts ℓ_B at $\beta = \frac{3}{4}$, and $15 = 120(1 - \gamma)$ puts ℓ_C at $\gamma = \frac{7}{8}$.

Along any line parallel to $\overline{BC}$, the coordinate β varies linearly, and on the chord at level α inside the triangle, β runs over an interval of length $1 - \alpha$ while the chord has length $220(1 - \alpha)$; hence a segment parallel to $\overline{BC}$ with endpoints differing by $\Delta\beta$ has length $220\,|\Delta\beta|$. The side of the new triangle on ℓ_A runs from $\ell_A \cap \ell_B$, where $\beta = \frac{3}{4}$, to $\ell_A \cap \ell_C$, where $\beta = 1 - \frac{3}{4} - \frac{7}{8} = -\frac{5}{8}$. Its length is

$$220\left(\frac{3}{4} + \frac{5}{8}\right) = 220 \cdot \frac{11}{8}.$$

Since the three lines are parallel to the sides of ABC, the triangle they bound is similar to ABC, here with ratio $\frac{11}{8}$. Its perimeter is $\frac{11}{8}(120 + 220 + 180) = \frac{11}{8} \cdot 520 = 715.$

8. The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and $f\left(\frac{1+\sqrt{3}i}{2}\right) = 2015 + 2019\sqrt{3}i$. Find the remainder when $f(1)$ is divided by 1000.

Solution:

Let $\omega = \frac{1+\sqrt{3}i}{2} = \cos 60^\circ + i\sin 60^\circ$, a primitive sixth root of unity. Since $2016 = 6 \cdot 336$, we get $\omega^{2016} = 1, \omega^{2017} = \omega$, and $\omega^{2018} = \omega^2 = \frac{-1+\sqrt{3}i}{2}$. Therefore

$$\begin{aligned} f(\omega) &= a\omega^2 + b\omega + c \\ &= \left(c + \frac{b-a}{2}\right) \\ &\quad + \frac{(a+b)\sqrt{3}}{2}\, i. \end{aligned}$$

Matching imaginary parts, $\frac{a+b}{2} = 2019$, so $a + b = 4038$. Since $a \leq 2019$ and $b \leq 2019$, this forces $a = b = 2019$. Matching real parts then gives $c + 0 = 2015$, so $c = 2015$.

Hence $f(1) = a + b + c = 4038 + 2015 = 6053$, whose remainder upon division by 1000 is 53.

9. Call a positive integer n k*-pretty* if n has exactly k positive divisors and n is divisible by k. For example, 18 is 6-pretty. Let S be the sum of the positive integers less than 2019 that are 20-pretty. Find $\frac{S}{20}$.

Solution:

We need $20 \mid n$ and $\tau(n) = 20$. Write $n = 2^a 5^b m$ with $\gcd(m, 10) = 1$; then $a \geq 2$, $b \geq 1$, and $(a+1)(b+1)\tau(m) = 20$ with $a+1 \geq 3$ and $b+1 \geq 2$. The factor $a+1$ must be a divisor of 20 that is at least $3:$ one of $4, 5, 10, 20.$

If $a+1 = 4$, then $(b+1)\tau(m) = 5$ forces $b = 4, m = 1$, so $n = 2^3 5^4 = 5000$, too large. If $a+1 = 10$, then $b = 1, m = 1$, and $n = 2^9 \cdot 5 = 2560$, too large. The case $a+1 = 20$ is impossible because $b+1 \geq 2$. If $a+1 = 5$, then $(b+1)\tau(m) = 4$, giving either $b = 3, m = 1$, so $n = 2^4 5^3 = 2000 < 2019$, or $b = 1$ and $\tau(m) = 2$, so $m = p$ is a prime other than 2 and 5 and $n = 80p < 2019$, i.e. $p \leq 25 : p \in \{3, 7, 11, 13, 17, 19, 23\}$.

Therefore

$$\begin{aligned} S &= 2000 \\ &\quad + 80_{(3+7+11+13+17+19+23)} \\ &= 2000 + 80 \cdot 93 \\ &= 9440, \end{aligned}$$

and $\frac{S}{20} = 472.$

10. There is a unique angle θ between 0° and 90° such that for nonnegative integers $n,$ the value of $\tan(2^n\theta)$ is positive when n is a multiple of $3,$ and negative otherwise. The degree measure of θ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q.$

Solution:

Since $\tan$ has period $180^\circ,$ only $2^n\theta \bmod 180^\circ$ matters: the tangent is positive on $(0^\circ, 90^\circ)$ and negative on $(90^\circ, 180^\circ).$ Suppose θ satisfies the condition, and let θ' be the reduction of 8θ modulo 180°; since $\tan(8\theta) > 0,$ we have $\theta' \in (0^\circ, 90^\circ).$ For every $n, 2^n\theta' \equiv 2^{n+3}\theta \pmod{180^\circ},$ and the sign pattern for the exponents $n+3$ is the same as for $n,$ so θ' also satisfies the condition. By uniqueness, $\theta' = \theta,$ so $7\theta \equiv 0 \pmod{180^\circ}$ and $\theta = \frac{180k}{7}$ degrees for some $k \in \{1, 2, 3\}.$

Test each: for $k = 1, 2\theta = \frac{360^\circ}{7} \approx 51.4^\circ$ has positive tangent — fails. For $k = 2, 4\theta = \frac{1440^\circ}{7} \equiv \frac{180^\circ}{7} \approx 25.7^\circ$ has positive tangent — fails. For $k = 3, \theta = \frac{540^\circ}{7} \approx 77.1^\circ$: then $2\theta \approx 154.3^\circ$ and $4\theta \equiv 128.6^\circ$ are both in $(90^\circ, 180^\circ),$ and $8\theta \equiv \theta,$ so the pattern positive, negative, negative repeats forever.

Thus $\theta = \frac{540}{7}$ degrees, and $p + q = 540 + 7 = 547.$

11. Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

By the tangent-chord angle in ω_1 (tangent AC, chord AK), $\angle KAC = \angle KBA$, and in ω_2 (tangent AB, chord AK), $\angle KAB = \angle KCA$. Write $u = \angle KAC$ and $v = \angle KAB$, so $u + v = \angle A$. Triangles KAB and KCA then have $\angle KAB = v = \angle KCA$ and $\angle KBA = u = \angle KAC$, so $\triangle KAB \sim \triangle KCA$. With $t = AK$ this gives

$$\frac{KB}{t} = \frac{t}{KC} = \frac{AB}{CA} = \frac{7}{9},$$

so $KB = \frac{7t}{9}$ and $KC = \frac{9t}{7}$. Also $\angle AKB = \angle AKC = 180^\circ - u - v = 180^\circ - \angle A$, so $\angle BKC = 360^\circ - 2(180^\circ - \angle A) = 2\angle A$.

From the law of cosines in ABC, $\cos A = \frac{49+81-64}{2\cdot 7\cdot 9} = \frac{11}{21}$, so $\cos 2A = 2\left(\frac{11}{21}\right)^2 - 1 = -\frac{199}{441}$. The law of cosines in triangle BKC gives

$$\begin{aligned} 64 &= \frac{49t^2}{81} + \frac{81t^2}{49} - 2t^2 \cos 2A \\ &= t^2 \cdot \frac{2401 + 6561 + 3582}{3969} \\ &= \frac{12544\, t^2}{3969}, \end{aligned}$$

so $t^2 = \frac{64\cdot 3969}{12544} = \frac{3969}{196}$ and $t = \frac{63}{14} = \frac{9}{2}$.

Hence $AK = \frac{9}{2}$ and $m + n = 9 + 2 = 11$.

12. For $n \geq 1$ call a finite sequence $(a_1, a_2, \ldots, a_n)$ of positive integers *progressive* if $a_i < a_{i+1}$ and a_i divides a_{i+1} for $1 \leq i \leq n-1$. Find the number of progressive sequences such that the sum of the terms in the sequence is equal to 360.

Solution:

Divisibility is transitive, so every term of a progressive sequence is a multiple of the first term. If a sequence with sum 360 has length at least 2 and first term d, then $d \mid 360$, and dividing the remaining terms by d yields a progressive sequence with first term at least 2 and sum $\frac{360-d}{d} = \frac{360}{d} - 1$; this correspondence is reversible. So if $g(s)$ denotes the number of progressive sequences with sum s and first term at least 2, the answer is $1 + \sum_{d|360,\, d<360} g\left(\frac{360}{d} - 1\right)$, the leading 1 counting the sequence (360).

The same reduction gives the recursion

$$g(s) = 1 + \sum_{\substack{e|s \\ 2 \leq e < s}} g\left(\frac{s}{e} - 1\right)$$
$$(s \geq 2),$$

with $g(1) = 0$; in particular $g(s) = 1$ when s is prime. Working upward: $g(2) = g(3) = g(4) = g(5) = g(7) = 1$; $g(6) = 1 + g(2) = 2$; $g(8) = 1 + g(3) = 2$; $g(9) = 1 + g(2) = 2$; $g(10) = 1 + g(4) = 2$; $g(12) = 1 + g(5) + g(3) + g(2) = 4$; $g(14) = 1 + g(6) = 3$; $g(16) = 1 + g(7) + g(3) = 3$; $g(21) = 1 + g(6) + g(2) = 4$; $g(35) = 1 + g(6) + g(4) = 4$; $g(39) = 1 + g(12) + g(2) = 6$; $g(44) = 1 + g(21) + g(10) + g(3) = 8$; $g(119) = 1 + g(16) + g(6) = 6$.

The 23 divisors $d < 360$ give arguments $\frac{360}{d} - 1 = 359, 179, 119, 89, 71, 59, 44, 39, 35, 29, 23, 19, 17, 14, 11, 9, 8, 7, 5, 4, 3, 2, 1$, whose g-values are $1, 1, 6, 1, 1, 1, 8, 6, 4, 1, 1, 1, 1, 3, 1, 2, 2, 1, 1, 1, 1, 1, 0$, summing to 46. Adding the single-term sequence gives $46 + 1 = 47$.

13. Regular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ is inscribed in a circle of area 1. Point P lies inside the circle so that the region bounded by $\overline{PA_1}, \overline{PA_2}$, and the minor arc $\widehat{A_1A_2}$ of the circle has area $\frac{1}{7}$, while the region bounded by $\overline{PA_3}, \overline{PA_4}$, and the minor arc $\widehat{A_3A_4}$ of the circle has area $\frac{1}{9}$. There is a positive integer n such that the area of the region bounded by $\overline{PA_6}, \overline{PA_7}$, and the minor arc $\widehat{A_6A_7}$ of the circle is equal to $\frac{1}{8} - \frac{\sqrt{2}}{n}$. Find n.

Solution:

Let O be the center, s the side length, a the apothem, and u_i the unit vector from O toward the midpoint of chord A_iA_{i+1}. The region bounded by $\overline{PA_i}, \overline{PA_{i+1}}$, and the arc is the circular segment together with triangle PA_iA_{i+1}. The segment has area $\frac{1}{8} - \frac{sa}{2}$ (sector minus triangle OA_iA_{i+1}), and the triangle at P has area $\frac{s}{2}(a - P \cdot u_i)$, since $a - P \cdot u_i$ is the distance from P to the chord. Adding,

$$\text{area}_i = \frac{1}{8} - \frac{s}{2}(P \cdot u_i).$$

The given areas say $\frac{s}{2}(P \cdot u_1) = \frac{1}{8} - \frac{1}{7} = -\frac{1}{56}$ and $\frac{s}{2}(P \cdot u_3) = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$. The normals rotate 45° per side, so u_3 is u_1 rotated 90°, and u_6, five steps from u_1, is u_1 rotated 225° :

$$u_6 = -\frac{\sqrt{2}}{2}(u_1 + u_3).$$

Therefore

$$\begin{aligned}
&\frac{s}{2}(P \cdot u_6) \\
&= -\frac{1}{\sqrt{2}}\left(-\frac{1}{56} + \frac{1}{72}\right) \\
&= -\frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{252}\right) \\
&= \frac{\sqrt{2}}{504}.
\end{aligned}$$

The region on side A_6A_7 thus has area $\frac{1}{8} - \frac{\sqrt{2}}{504}$, so $n = 504.$

14. Find the sum of all positive integers n such that, given an unlimited supply of stamps of denominations $5, n,$ and $n+1$ cents, 91 cents is the greatest postage that cannot be formed.

Solution:

Using k stamps of the denominations n and $n+1$ produces exactly the amounts $kn + c$ for $0 \leq c \leq k$, and adding 5-cent stamps then covers everything above in the same residue class mod 5. So in each class r every amount at least $m(r)$ is formable and nothing smaller is, where $m(r)$ is the least value of $kn + c\,(0 \leq c \leq k)$ congruent to r mod 5. The greatest non-formable amount is $\max_r m(r) - 5$, so we need $\max_r m(r) = 96$: the class of 96 (which is 1 mod 5) must be covered first exactly at 96, and every other class no later.

Case on $n \bmod 5$, noting $kn + c \equiv kn + c$. If $n \equiv 4$: class 1 needs $4k + c \equiv 1$ with $c \leq k$, first possible at $k = 4, c = 0$, so $4n = 96$ and $n = 24$; the other classes are covered at $24, 48, 72$, all less than 96, so $n = 24$ works. If $n \equiv 2$: class 1 is first covered at $k = 2, c = 2$, so $2n + 2 = 96$ and $n = 47$; the other classes are covered at $47, 48, 94 \leq 96$, so $n = 47$ works.

If $n \equiv 3$: class 1 first at $2n = 96$, so $n = 48$, but then class 2 is first covered at $2n + 1 = 97 > 96$ – fails. If $n \equiv 1$: class 1 first at $n = 96$, but then class 3 is first covered at $2n + 1 = 193$ – fails. If $n \equiv 0$: class 1 first at $n + 1 = 96$, so $n = 95$, but class 4 needs $c = 4, k \geq 4$, giving $4n + 4 > 96$ – fails. The answer is $24 + 47 = 71.$

15. In acute triangle ABC, points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

Solution:

Write $b = AC$, $c = AB$, $a = BC$, and $k = \cos A$. Right triangles APC and AQB give $AP = bk$ and $AQ = ck$, so triangle APQ is similar to triangle ACB with ratio k, whence $PQ = ak = 25$. The points on the line occur in the order X, P, Q, Y, so the power of P gives $XP \cdot PY = 10 \cdot 40 = 400 = AP \cdot PB$, and the power of Q gives $YQ \cdot QX = 15 \cdot 35 = 525 = AQ \cdot QC$. With $u = bk$ and $v = ck$ these read

$$\begin{aligned} u(c-u) &= 400, \\ v(b-v) &= 525, \end{aligned}$$

that is, $w - u^2 = 400$ and $w - v^2 = 525$, where $w = \frac{uv}{k} = bck$.

By the law of cosines, $a^2 = b^2 + c^2 - 2bck$, so $a^2k^2 = u^2 + v^2 - 2uvk = 625$. Substituting $u^2 = w - 400$ and $v^2 = w - 525$, and $uv = wk$, gives $2w - 925 - 2wk^2 = 625$, so $wk^2 = w - 775$. Then

$$\begin{aligned} (uv)^2 &= w^2k^2 \\ &= w(w-775) \\ &= (w-400)(w-525), \end{aligned}$$

which simplifies to $150w = 210000$, so $w = 1400$ and $k^2 = \frac{1400-775}{1400} = \frac{25}{56}$.

Thus $k = \frac{5}{2\sqrt{14}}$ and

$$\begin{aligned} bc = \frac{w}{k} &= 1400 \cdot \frac{2\sqrt{14}}{5} \\ &= 560\sqrt{14}, \end{aligned}$$

so $m + n = 560 + 14 = 574$.

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