2020 AIME II Solutions

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1. Find the number of ordered pairs of positive integers (m, n) such that $m^2 n = 20^{20}$.

Solution:

Since $20^{20} = 2^{40} \cdot 5^{20}$, a valid m must have the form $2^a 5^b$, and then $n = 2^{40-2a}\,5^{20-2b}$ is a positive integer exactly when $2a \le 40$ and $2b \le 20$. Conversely every such choice works, and n is uniquely determined by m.

So a can be any of $0, 1, \ldots, 20$ and b any of $0, 1, \ldots, 10$, giving $21 \cdot 11 = 231$ ordered pairs.

2. Let P be a point chosen uniformly at random in the interior of the unit square with vertices at $(0,0),(1,0),(1,1)$, and $(0,1)$. The probability that the slope of the line determined by P and the point $\left(\frac{5}{8},\frac{3}{8}\right)$ is greater than or equal to $\frac{1}{2}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Let $Q=\left(\frac{5}{8},\frac{3}{8}\right)$ and $P=(x,y)$. The slope condition $\frac{y-\frac{3}{8}}{x-\frac{5}{8}}\geq\frac{1}{2}$ becomes $y\geq\frac{x}{2}+\frac{1}{16}$ when $x>\frac{5}{8}$, and $y\leq\frac{x}{2}+\frac{1}{16}$ when $x<\frac{5}{8}$ (multiplying by the negative quantity $x-\frac{5}{8}$ reverses the inequality).

For $x>\frac{5}{8}$, the region above the line $y=\frac{x}{2}+\frac{1}{16}$ inside the square is a trapezoid with parallel vertical sides of lengths $\frac{5}{8}$ (at $x=\frac{5}{8}$) and $\frac{7}{16}$ (at $x=1$) and width $\frac{3}{8}$, with area $\frac{3}{8}\cdot\frac{\frac{5}{8}+\frac{7}{16}}{2}=\frac{51}{256}$. For $x<\frac{5}{8}$, the region below the line is a trapezoid with parallel sides $\frac{1}{16}$ (at $x=0$) and $\frac{3}{8}$ (at $x=\frac{5}{8}$) and width $\frac{5}{8}$, with area $\frac{5}{8}\cdot\frac{\frac{1}{16}+\frac{3}{8}}{2}=\frac{35}{256}$.

The probability is $\frac{51}{256}+\frac{35}{256}=\frac{86}{256}=\frac{43}{128}$, so $m+n=43+128=171$.

3. The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

By the change-of-base formula,

$$\log_{2^x} 3^{20} = \frac{20 \log 3}{x \log 2}$$

and

$$\log_{2^{x+3}} 3^{2020} = \frac{2020 \log 3}{(x+3) \log 2}.$$

Cancelling the common factor $\frac{\log 3}{\log 2}$ leaves $\frac{20}{x} = \frac{2020}{x+3}$.

Cross-multiplying gives $20x + 60 = 2020x$, so $2000x = 60$ and $x = \frac{3}{100}$. Thus $m + n = 3 + 100 = 103$.

4. Triangles $\triangle ABC$ and $\triangle A'B'C'$ lie in the coordinate plane with vertices $A(0,0)$, $B(0,12)$, $C(16,0)$, $A'(24,18)$, $B'(36,18)$, $C'(24,2)$. A rotation of m degrees clockwise around the point (x,y), where $0<m<180$, will transform $\triangle ABC$ to $\triangle A'B'C'$. Find $m+x+y$.

Solution:

The vector $\overrightarrow{AB}=(0,12)$ is vertical, while $\overrightarrow{A'B'}=(12,0)$ is horizontal and of the same length, so the rotation turns directions by 90° clockwise, and $m=90$.

A 90° clockwise rotation about (a,b) sends (p,q) to $(a+q-b,\ b-p+a)$. Applying this to $A=(0,0)$ and setting the image equal to $A'=(24,18)$ gives $a-b=24$ and $a+b=18$, so $a=21$ and $b=-3$. Checking the other vertices: $B=(0,12)$ maps to $(21+12+3,\ -3+21)=(36,18)=B'$, and $C=(16,0)$ maps to $(21+3,\ -3-16+21)=(24,2)=C'$.

Therefore $m+x+y=90+21+(-3)=108$.

5. For each positive integer n, let $f(n)$ be the sum of the digits in the base-four representation of n and let $g(n)$ be the sum of the digits in the base-eight representation of $f(n)$. For example, $f(2020) = f(133210_4) = 10 = 12_8$, and the digit sum of 12_8 is 3, so $g(2020) = 3$. Let N be the least value of n such that the base-sixteen representation of $g(n)$ cannot be expressed using only the digits 0 through 9. Find the remainder when N is divided by 1000.

Solution:

The base-sixteen representation of $g(n)$ needs a digit beyond 9 exactly when $g(n) \geq 10$. So we need the base-eight digit sum of $f(n)$ to be at least 10. Checking values in order, every number less than 31 has base-eight digit sum at most 9, while $31 = 37_8$ has digit sum 10. Since the least n achieving a given base-four digit sum increases with that sum, we want the least n with $f(n) = 31$.

A base-four digit is at most 3, so a digit sum of 31 requires at least 11 digits, and the smallest 11-digit choice is a leading 1 followed by ten 3s:

$$\begin{aligned} N &= 13333333333_4 \\ &= 4^{10} + (4^{10} - 1) \\ &= 2 \cdot 4^{10} - 1 = 2097151. \end{aligned}$$

The remainder when $N = 2097151$ is divided by 1000 is 151.

6. Define a sequence recursively by $t_1 = 20, t_2 = 21,$ and

$$t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$$

for all $n \geq 3.$ Then t_{2020} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q.$

Solution:

Computing terms exactly:

$$\begin{aligned} t_3 &= \frac{5 \cdot 21 + 1}{25 \cdot 20} = \frac{53}{250}, \\ t_4 &= \frac{5 \cdot \frac{53}{250} + 1}{25 \cdot 21} = \frac{103}{26250}, \\ t_5 &= \frac{5 \cdot \frac{103}{26250} + 1}{25 \cdot \frac{53}{250}} \\ &= \frac{5353}{5250} \cdot \frac{10}{53} = \frac{101}{525}, \end{aligned}$$

using $5353 = 53 \cdot 101.$ Then $t_6 = \frac{5 \cdot \frac{101}{525} + 1}{25 \cdot \frac{103}{26250}} = \frac{206}{105} \cdot \frac{1050}{103} = 20 = t_1$ and $t_7 = \frac{101}{\frac{101}{21}} = 21 = t_2.$

Since each term depends only on the two preceding terms, the sequence repeats with period $5.$ Because 2020 is a multiple of $5,$ we get $t_{2020} = t_5 = \frac{101}{525}.$ As 101 is prime and does not divide $525,$ the fraction is reduced, and $p + q = 101 + 525 = 626.$

7. Two congruent right circular cones each with base radius 3 and height 8 have axes of symmetry that intersect at right angles at a point in the interior of the cones a distance 3 from the base of each cone. A sphere with radius r lies within both cones. The maximum possible value of r^2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Put the origin at the point where the axes cross, and measure signed coordinates u_1, u_2 along the two axes. Along its axis, each cone has its apex at distance $8-3=5$ from the origin and its base plane at distance 3 on the other side. Slicing a cone by any plane through its axis gives a triangle whose slant side, in coordinates (u,w) with $w \geq 0$ the distance from the axis, is the line through $(5,0)$ and $(-3,3)$, namely $3u+8w=15$.

A sphere of radius r centered at a point with axial coordinate u and distance ρ from the axis fits inside that cone only if its cross-section fits inside the triangle, so $r \leq \frac{15-3u-8\rho}{\sqrt{73}}$. Since the two axes are perpendicular, the distance from the center to axis 1 is at least $|u_2|$, and vice versa. Adding the two constraints,

$$\begin{aligned} 2\sqrt{73}\,r &\leq 30-3(u_1+u_2) \\ &\quad -8(|u_1|+|u_2|) \\ &\leq 30, \end{aligned}$$

because $3(u_1+u_2) \geq -3(|u_1|+|u_2|) \geq -8(|u_1|+|u_2|)$. Hence $r \leq \frac{15}{\sqrt{73}}$.

The sphere of radius $\frac{15}{\sqrt{73}}$ centered at the origin achieves this: its distance to each slant surface is $\frac{|3\cdot 0+8\cdot 0-15|}{\sqrt{3^2+8^2}} = \frac{15}{\sqrt{73}}$, and its distance 3 to each base plane is larger. So the maximum of r^2 is $\frac{225}{73}$, and $m+n=225+73=298$.

8. Define a sequence recursively by $f_1(x) = |x-1|$ and $f_n(x) = f_{n-1}(|x-n|)$ for integers $n > 1$. Find the least value of n such that the sum of the zeros of f_n exceeds $500{,}000$.

Solution:

Since $f_n(x) = f_{n-1}(|x-n|)$, the zeros of f_n are exactly $x = n \pm z$ as z runs over the nonnegative zeros of f_{n-1}. The first few zero sets are $\{1\}, \{1, 3\}, \{0, 2, 4, 6\}$, $\{-2, 0, 2, \ldots, 10\}$. Writing $T_k = \frac{k(k+1)}{2}$, we claim the zeros of f_n are every other integer from $n - T_{n-1}$ through T_n. Indeed, by induction the nonnegative zeros of f_{n-1} are every other integer from 0 or 1 up to T_{n-1}, and applying $n \pm z$ yields every integer of the appropriate parity from $n - T_{n-1}$ through $n + T_{n-1} = T_n$.

This progression has $\frac{T_n + T_{n-1} - n}{2} + 1 = \frac{n(n-1)}{2} + 1$ terms, and its first and last terms sum to $(n - T_{n-1}) + T_n = 2n$, so the sum of the zeros is

$$S_n = n\left(\frac{n(n-1)}{2} + 1\right),$$

which is increasing in n.

Now $S_{100} = 100 \cdot 4951 = 495{,}100 \leq 500{,}000$, while $S_{101} = 101 \cdot 5051 = 510{,}151 > 500{,}000$. The least such n is 101.

9. While watching a show, Ayako, Billy, Carlos, Dahlia, Ehuang, and Frank sat in that order in a row of six chairs. During the break, they went to the kitchen for a snack. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

Solution:

Number the friends 1 through 6 in original seating order; we count orderings of $1, \ldots, 6$ in which no two consecutive integers are adjacent. Apply inclusion-exclusion over which of the five pairs $\{i, i+1\}$ are forced to sit together. If a chosen set of k pairs forms r maximal runs of consecutive integers, gluing each run into a block (orderable ascending or descending) gives $2^r(6-k)!$ seatings containing all chosen adjacencies.

Tallying by k : for $k=0, 720$. For $k=1$, five sets, each $2 \cdot 120$, total 1200. For $k=2$, four sets form one run $(2 \cdot 24)$ and six form two runs $(4 \cdot 24)$, total 768. For $k=3$, three sets form one run $(2 \cdot 6)$, six form two runs $(4 \cdot 6)$, one forms three runs $(8 \cdot 6)$, total 228. For $k=4$, two sets form one run $(2 \cdot 2)$ and three form two runs $(4 \cdot 2)$, total 32. For $k=5$, one set, total 2.

The count is $720-1200+768-228+32-2=90$.

10. Find the sum of all positive integers n such that when $1^3 + 2^3 + 3^3 + \cdots + n^3$ is divided by $n + 5$, the remainder is 17.

Solution:

Let $K = \frac{n(n+1)}{2}$, so the sum of cubes is K^2. Modulo $n + 5$ we have $n \equiv -5$, hence $2K = n(n + 1) \equiv (-5)(-4) = 20$ and $4K^2 \equiv 400$. If $K^2 \equiv 17 \pmod{n + 5}$, then $n + 5$ divides $400 - 4 \cdot 17 = 332 = 2^2 \cdot 83$. Since a remainder of 17 also forces $n + 5 > 17$, the candidates are $n + 5 \in \{83, 166, 332\}$, i.e. $n \in \{78, 161, 327\}$.

Because we multiplied by 4, each candidate must be checked. For $n = 78 : K = 3081 \equiv 10 \pmod{83}$, and $10^2 = 100 \equiv 17 \pmod{83}$. For $n = 161 : K = 13041 \equiv 93 \pmod{166}$, and $93^2 = 8649 = 52 \cdot 166 + 17$. For $n = 327 : K = 53628 \equiv 176 \pmod{332}$, and $176^2 = 30976 = 93 \cdot 332 + 100$, remainder $100 \neq 17$, so this one fails.

The valid values are $n = 78$ and $n = 161$, with sum $78 + 161 = 239$.

11. Let $P(x)=x^2-3x-7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P+Q$, $P+R$, and $Q+R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0)=2$, then $R(0)=\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Let a be the common root of $P+Q$ and $P+R$, let b be that of $P+Q$ and $Q+R$, and let c be that of $P+R$ and $Q+R$. Each sum is quadratic with leading coefficient 2, so $P+Q=2(x-a)(x-b)$, $P+R=2(x-a)(x-c)$, and $Q+R=2(x-b)(x-c)$. Write $Q=x^2+qx+2$ and $R=x^2+rx+s$. By Vieta's formulas, the root sums are $a+b=\frac{3-q}{2}$, $a+c=\frac{3-r}{2}$, and $b+c=-\frac{q+r}{2}$, so

$$\begin{aligned}2a&=(a+b)+(a+c)-(b+c)\\&=\frac{3-q}{2}+\frac{3-r}{2}\\&\quad+\frac{q+r}{2}=3,\end{aligned}$$

giving $a=\frac{3}{2}$.

The constant term of $P+Q$ is $-7+2=-5$, so $2ab=-5$ and $b=-\frac{5}{3}$. Also $2ac=P(0)+R(0)=s-7$ and $2bc=Q(0)+R(0)=s+2$. Subtracting, $2c(b-a)=9$, so $c=\frac{9}{2\left(-\frac{5}{3}-\frac{3}{2}\right)}=-\frac{27}{19}$.

Then $s=2bc-2=2\cdot\left(-\frac{5}{3}\right)\left(-\frac{27}{19}\right)-2=\frac{90}{19}-\frac{38}{19}=\frac{52}{19}$, so $R(0)=\frac{52}{19}$ and $m+n=52+19=71$.

12. Let m and n be odd integers greater than 1. An $m \times n$ rectangle is made up of unit squares where the squares in the top row are numbered left to right with the integers 1 through n, those in the second row are numbered left to right with the integers $n+1$ through $2n$, and so on. Square 200 is in the top row, and square 2000 is in the bottom row. Find the number of ordered pairs (m, n) of odd integers greater than 1 with the property that, in the $m \times n$ rectangle, the line through the centers of squares 200 and 2000 intersects the interior of square 1099.

Solution:

Use coordinates $(\text{column}, \text{row})$. Square 200 is in the top row, so $n \geq 200$ (hence $n \geq 201$ as n is odd) and its center is $(200, 1)$. Square 2000 is in the bottom row, so its column is $b = 2000 - (m-1)n$ with $1 \leq b \leq n$, i.e. $(m-1)n < 2000 \leq mn$, and its center is (b, m). Since m and n are odd, b is even, so the midpoint $\left(\frac{200+b}{2}, \frac{m+1}{2}\right)$ of the two centers has integer coordinates; its square number is $\frac{(m-1)n+200+b}{2} = \frac{2200}{2} = 1100$. Its column $100 + \frac{b}{2}$ lies between 101 and n, so the line passes through the center of square 1100, and square 1099 sits immediately to its left in the same row.

Square 1099 lies only in that row, and the line crosses that row's horizontal strip in a segment centered (by symmetry) at the center of square 1100, extending $\frac{1}{2|s|}$ to each side, where $s = \frac{m-1}{b-200}$ is the slope. So the line meets the interior of square 1099 exactly when $\frac{1}{2|s|} > \frac{1}{2}$, that is $|s| < 1$, i.e. $m - 1 < |b - 200| = |1800 - (m-1)n|$ (a vertical line, $b = 200$, fails).

Since $n \geq 201$ and $(m-1)n < 2000$, we need $m - 1 < 10$, so $m \in \{3, 5, 7, 9\}$. For $m = 3:$ odd $n \in [667, 999]$, 167 values, excluding $|n - 900| \leq 1$ (odd cases $899, 901$) leaves 165. For $m = 5:$ odd $n \in [401, 499]$, 50 values, excluding $449, 451$ leaves 48. For $m = 7:$ odd $n \in [287, 333]$, 24 values, excluding $299, 301$ leaves 22. For $m = 9:$ odd $n \in [223, 249]$, 14 values, excluding 225 leaves 13. The total is $165 + 48 + 22 + 13 = 248$.

13. Convex pentagon $ABCDE$ has side lengths $AB = 5, BC = CD = DE = 6$, and $EA = 7$. Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of $ABCDE$.

Solution:

Let the tangent lengths from A, B, C, D, E to the incircle be a, b, c, d, e. Then $a + b = 5, b + c = 6, c + d = 6, d + e = 6$, and $e + a = 7$. The middle equations give $d = b$ and $e = c$, so $c + a = 7$; with $a + b = 5$ and $b + c = 6$ this yields $a = 3, b = d = 2, c = e = 4$. If r is the inradius, the interior angle at a vertex with tangent length t satisfies $\tan \frac{\theta}{2} = \frac{r}{t}$, and the half-angles sum to half of 540° :

$$\arctan \frac{r}{3} + 2 \arctan \frac{r}{2} \\ + 2 \arctan \frac{r}{4} = 270^\circ.$$

Let $\beta = \arctan \frac{r}{2}$ and $\gamma = \arctan \frac{r}{4}$. Then $\arctan \frac{r}{3} = 270^\circ - 2(\beta + \gamma)$, and since $\tan(270^\circ - \theta) = \cot \theta$, we get $\frac{r}{3} = \cot 2(\beta + \gamma)$. With $T = \tan(\beta + \gamma) = \frac{\frac{r}{2} + \frac{r}{4}}{1 - \frac{r^2}{8}} = \frac{6r}{8 - r^2}$, the identity $\frac{r}{3} = \frac{1 - T^2}{2T}$ becomes $2rT = 3(1 - T^2)$, and substituting T and clearing denominators gives

$$5r^4 - 84r^2 + 64 = 0,$$

so $r^2 = 16$ or $r^2 = \frac{4}{5}$. For $r^2 = \frac{4}{5}$ every half-angle is well under 54°, so the half-angle sum falls far short of 270°; this root is extraneous. Hence $r = 4$.

The semiperimeter is $s = \frac{5+6+6+6+7}{2} = 15$, so the area is $rs = 4 \cdot 15 = 60$.

14. For real number x let $\lfloor x \rfloor$ be the greatest integer less than or equal to x, and define $\{x\} = x - \lfloor x \rfloor$ to be the fractional part of x. For example, $\{3\} = 0$ and $\{4.56\} = 0.56$. Define $f(x) = x\{x\}$, and let N be the number of real-valued solutions to the equation $f(f(f(x))) = 17$ for $0 \le x \le 2020$. Find the remainder when N is divided by 1000.

Solution:

On $[k, k+1)$ with $k \ge 0$ an integer, write $x = k + t$; then $f(x) = (k+t)t$ is strictly increasing from 0 toward $k+1$, so f maps $[k, k+1)$ bijectively onto $[0, k+1)$. Hence for any c with $n < c < n+1$, the equation $f(w) = c$ has exactly one solution in $[k, k+1)$ for each integer $k \ge n$, and no others.

The equation $f(z) = 17$ has one solution $z_n \in (n, n+1)$ for each $n \ge 17$. In turn, $f(w) = z_n$ has one solution in $(k, k+1)$ for each $k \ge n$. Finally, for $x \in [j, j+1)$ with $0 \le j \le 2019$, f maps $[j, j+1)$ bijectively onto $[0, j+1)$, so the number of solutions of $f(f(f(x))) = 17$ there equals the number of such w with $w < j+1$, namely the number of pairs (n, k) with $17 \le n \le k \le j$, which is $\binom{j-15}{2}$. (The endpoint $x = 2020$ gives $f(x) = 0$ and is not a solution.)

By the hockey stick identity,

$$\begin{aligned} N &= \sum_{j=17}^{2019} \binom{j-15}{2} \\ &= \sum_{a=2}^{2004} \binom{a}{2} \\ &= \binom{2005}{3} \\ &= \frac{2005 \cdot 2004 \cdot 2003}{6} \\ &= 1{,}341{,}349{,}010, \end{aligned}$$

so the remainder when N is divided by 1000 is 10.

15. Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT = CT = 16, BC = 22$, and $TX^2 + TY^2 + XY^2 = 1143$. Find XY^2.

Solution:

By the tangent-chord angle, $\angle TBC = A$, so $\angle ABT = B + A = 180^\circ - C$ and $TX = TB \sin \angle ABT = 16 \sin C$; similarly $TY = 16 \sin B$. Also $\angle AXT = \angle AYT = 90^\circ$, so A, X, T, Y lie on a circle with diameter AT, whence $XY = AT \sin A$. Using the law of sines $\left(\sin A = \frac{11}{R},\ \sin B = \frac{AC}{2R},\ \sin C = \frac{AB}{2R}\right)$, the given condition becomes

$$\frac{64\left(AB^2 + AC^2\right) + 121\, AT^2}{R^2} = 1143.$$

Place $B = (-11, 0)$ and $C = (11, 0)$. Since $TB = 16$ and T lies on the perpendicular bisector of BC, we get $T = (0, -\sqrt{135})$. The circumcenter is $O = (0, k)$ with $OB \perp BT$, which gives $121 - k\sqrt{135} = 0$, so $k = \frac{121}{\sqrt{135}}$ and $R^2 = 121 + k^2 = \frac{30976}{135}$. For $A = (x, y)$ on ω, expanding $x^2 + (y-k)^2 = R^2$ gives $x^2 + y^2 = \frac{242}{\sqrt{135}}\, y + 121$. Therefore

$$\begin{aligned} AB^2 + AC^2 &= 2(x^2 + y^2) + 242 \\ &= \frac{484}{\sqrt{135}}\, y + 484, \end{aligned}$$

$$\begin{aligned} AT^2 &= x^2 + y^2 \\ &\quad + 2\sqrt{135}\, y + 135 \\ &= \frac{512}{\sqrt{135}}\, y + 256. \end{aligned}$$

Substituting, $64(AB^2 + AC^2) + 121\, AT^2 = \frac{92928}{\sqrt{135}}\, y + 61952 = 1143 \cdot \frac{30976}{135}$ yields $y = \frac{291}{\sqrt{135}}$. Then $AT^2 = \frac{512 \cdot 291}{135} + 256 = \frac{183552}{135}$, and

$$
\begin{aligned}
XY^2 &= AT^2 \sin^2 A \\
&= \frac{121\, AT^2}{R^2} \\
&= \frac{121 \cdot 183552}{30976} \\
&= \frac{183552}{256} = 717.
\end{aligned}
$$

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