2021 AIME I Solutions

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1. Zou and Chou are practicing their 100-meter sprints by running 6 races against each other. Zou wins the first race, and after that, the probability that one of them wins a race is $\frac{2}{3}$ if they won the previous race but only $\frac{1}{3}$ if they lost the previous race. The probability that Zou will win exactly 5 of the 6 races is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Zou wins race 1, so winning exactly 5 of the 6 races means he loses exactly one of races 2 through 6. Each race after the first repeats the previous outcome with probability $\frac{2}{3}$ and switches with probability $\frac{1}{3}$.

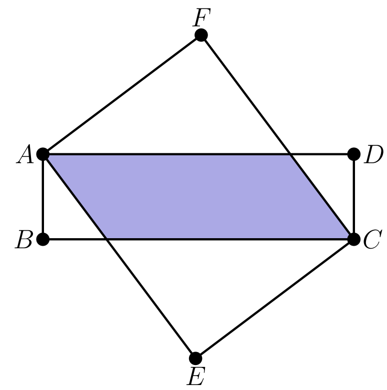
If the loss is race 6, the five transitions are four repeats followed by one switch: $\left(\frac{2}{3}\right)^4 \cdot \frac{1}{3} = \frac{16}{243}$. If the loss is race i for some $2 \le i \le 5$, there is a switch into the loss and a switch back to winning, plus three repeats: $\left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = \frac{8}{243}$ for each of the 4 positions, contributing $\frac{32}{243}$.

The total is

$$\frac{16}{243} + \frac{32}{243} = \frac{48}{243} = \frac{16}{81},$$

so $m+n = 16+81 = 97$.

2. In the diagram below, $ABCD$ is a rectangle with side lengths $AB = 3$ and $BC = 11$, and $AECF$ is a rectangle with side lengths $AF = 7$ and $FC = 9$, as shown. The area of the shaded region common to the interiors of both rectangles is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution:

Place $B = (0, 0), C = (11, 0), A = (0, 3), D = (11, 3)$. Solving $AF = 7$ and $CF = 9$ (consistent since $AC^2 = 11^2 + 3^2 = 130 = 7^2 + 9^2$) gives $F = \left(\frac{28}{5}, \frac{36}{5}\right)$, and since the diagonals of rectangle $AECF$ bisect each other, $E = A + C - F = \left(\frac{27}{5}, -\frac{21}{5}\right)$.

Sides AE and FC have direction $(3, -4)$, lying on the lines $4x + 3y = 9$ and $4x + 3y = 44$; sides AF and EC lie on $3x - 4y = -12$ and $3x - 4y = 33$. Every point of $ABCD$ satisfies $-12 \leq 3x - 4y \leq 33$, so the common region is just the part of the strip $0 \leq y \leq 3$ between the lines $4x + 3y = 9$ and $4x + 3y = 44$: a parallelogram with vertices $A = (0, 3)$, $\left(\frac{9}{4}, 0\right)$, $C = (11, 0)$, and $\left(\frac{35}{4}, 3\right)$.

Its horizontal sides have length $11 - \frac{9}{4} = \frac{35}{4}$ and the height between them is 3, so the area is $\frac{35}{4} \cdot 3 = \frac{105}{4}$, and $m + n = 105 + 4 = 109$.

3. Find the number of positive integers less than 1000 that can be expressed as the difference of two integral powers of 2.

Solution:

A difference of powers of 2 is $2^a - 2^b = 2^b(2^c - 1)$ where $c = a - b \geq 1$. Since $2^c - 1$ is odd, the odd part of the number determines c and the power of 2 determines b, so distinct pairs (b, c) yield distinct integers. It suffices to count pairs with $2^b(2^c - 1) < 1000$.

For $c = 1, 2, \ldots, 9$ the factor $2^c - 1$ is $1, 3, 7, 15, 31, 63, 127, 255, 511$, and the number of allowed values of b is $10, 9, 8, 7, 6, 4, 3, 2, 1$ respectively (the count for 63 drops to 4 because $63 \cdot 16 = 1008 > 1000$, while $31 \cdot 32 = 992$ still fits).

The total is $10 + 9 + 8 + 7 + 6 + 4 + 3 + 2 + 1 = 50$.

4. Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

Solution:

The ordered triples (a, b, c) of positive integers with $a + b + c = 66$ number $\binom{65}{2} = 2080$. Exactly one of them has all three values equal, namely $(22, 22, 22)$. Triples with exactly two values equal come from $2a + c = 66$ with $c \neq a$: here a can be 1 through 32 except 22, giving 31 multisets, each arrangeable in 3 ways, so 93 ordered triples.

Hence $2080 - 1 - 93 = 1986$ ordered triples have three distinct values, and each unordered choice $a < b < c$ is counted 6 times. The number of valid separations is $\frac{1986}{6} = 331$.

5. Call a three-term strictly increasing arithmetic sequence of integers *special* if the sum of the squares of the three terms equals the product of the middle term and the square of the common difference. Find the sum of the third terms of all special sequences.

Solution:

Write the terms as $a-d, a, a+d$ with integer $d \geq 1$. The condition is

$$\begin{aligned} &(a-d)^2+a^2 \\ &\quad +(a+d)^2=ad^2 \\ &\quad \Longleftrightarrow \quad 3a^2+2d^2=ad^2, \end{aligned}$$

so $d^2(a-2)=3a^2$ and $d^2=\frac{3a^2}{a-2}$. For d^2 to be positive we need $a>2$. Indeed, $a=0$ forces $d=0$, while for any nonzero integer $a<2$ the right side is negative; $a=2$ makes the original equation impossible.

Substituting $t=a-2 \geq 1$ gives

$$d^2=\frac{3(t+2)^2}{t}=3t+12+\frac{12}{t},$$

so $t \mid 12$. Testing $t=1,2,3,4,6,12$ gives $d^2=27,24,25,27,32,49:$ only $t=3$ and $t=12$ yield perfect squares.

These give $(a,d)=(5,5)$ with sequence $0,5,10$, and $(a,d)=(14,7)$ with sequence $7,14,21$. The sum of the third terms is $10+21=31$.

6. Segments $\overline{AB}, \overline{AC}$, and $\overline{AD}$ are edges of a cube and $\overline{AG}$ is a diagonal through the center of the cube. Point P satisfies $BP = 60\sqrt{10}, CP = 60\sqrt{5}, DP = 120\sqrt{2}$, and $GP = 36\sqrt{7}$. Find AP.

Solution:

Let A be the origin with $B = (s, 0, 0), C = (0, s, 0), D = (0, 0, s), G = (s, s, s)$, and $P = (x, y, z)$. Expanding,

$$\begin{aligned} BP^2 &= AP^2 - 2sx + s^2, \\ CP^2 &= AP^2 - 2sy + s^2, \\ DP^2 &= AP^2 - 2sz + s^2, \end{aligned}$$

while $GP^2 = AP^2 - 2s(x + y + z) + 3s^2$. Therefore

$$\begin{aligned} &BP^2 + CP^2 \\ &\quad + DP^2 - GP^2 \\ &= 2\,AP^2, \end{aligned}$$

with every term involving s or the coordinates of P cancelling.

The given lengths yield $BP^2 = 36000, CP^2 = 18000, DP^2 = 28800$, and $GP^2 = 9072$, so

$$\begin{aligned} 2\,AP^2 &= 36000 + 18000 \\ &\quad + 28800 - 9072 \\ &= 73728, \end{aligned}$$

giving $AP^2 = 36864$ and $AP = 192$.

7. Find the number of pairs (m, n) of positive integers with $1 \leq m < n \leq 30$ such that there exists a real number x satisfying

$$\sin(mx) + \sin(nx) = 2.$$

Solution:

Since each sine is at most $1,$ we need $\sin(mx) = \sin(nx) = 1,$ i.e. $mx = \frac{\pi}{2} + 2\pi a$ and $nx = \frac{\pi}{2} + 2\pi b$ for integers $a, b.$ Eliminating x gives $n(4a + 1) = m(4b + 1),$ that is, $4(na - mb) = m - n.$ As a and b range over the integers, $na - mb$ takes exactly the multiples of $g = \gcd(m, n),$ so a solution exists if and only if $4g \mid n - m$ — equivalently, writing $m = gm'$ and $n = gn'$, if and only if $m' \equiv n' \pmod 4$ (which forces both m' and n' odd).

For each g we count coprime pairs $m' < n' \leq \lfloor \frac{30}{g} \rfloor$ of odd numbers in the same class mod $4.$ For $g = 1$: among $1, 3, \ldots, 29$ there are eight numbers $\equiv 1$ and seven $\equiv 3 \pmod 4$, giving $\binom{8}{2} + \binom{7}{2} = 49$ pairs, of which the five pairs $\{3, 15\}$, $\{3, 27\}$, $\{9, 21\}$, $\{15, 27\}$, $\{5, 25\}$ are not coprime, leaving $44.$ For $g = 2$ (odd numbers up to 15): $\binom{4}{2} + \binom{4}{2} = 12$ minus the pair $\{3, 15\}$ gives $11.$ For $g = 3$ (up to 10): the pairs $\{1, 5\}$, $\{1, 9\}$, $\{5, 9\}$, $\{3, 7\}$ give $4.$ For $g = 4$ (up to 7): $\{1, 5\}$ and $\{3, 7\}$ give $2.$ For $g = 5$ and $g = 6$: only $\{1, 5\}$, giving 1 each. For $g \geq 7$ we would need two distinct odd numbers up to $\lfloor \frac{30}{g} \rfloor \leq 4$ in the same class mod $4,$ which is impossible.

The total is $44 + 11 + 4 + 2 + 1 + 1 = 63.$

8. Find the number of integers c such that the equation

$$\left| \left| 20|x| - x^2 \right| - c \right| = 21$$

has 12 distinct real solutions.

Solution:

Set $f(x) = \left|20|x| - x^2\right|$, an even function; the equation says $f(x) = c + 21$ or $f(x) = c - 21$. For $x \geq 0$, the graph of f rises from 0 to 100 on $[0, 10]$, falls back to 0 at $x = 20$, then increases without bound. So for k with $0 < k < 100$, the equation $f(x) = k$ has 3 positive solutions, hence 6 solutions in all; for $k = 100$ it has 4; for $k > 100$ it has 2; and for $k = 0$ it has 3 (namely 0 and ± 20).

The two levels $c + 21$ and $c - 21$ are distinct, so the only way to reach 12 solutions is $6 + 6$: both $c - 21$ and $c + 21$ must lie strictly between 0 and 100. This means $c \geq 22$ and $c \leq 78$, and every such integer works: there are $78 - 22 + 1 = 57$ values.

9. Let $ABCD$ be an isosceles trapezoid with $AD = BC$ and $AB < CD$. Suppose that the distances from A to the lines BC, CD, and BD are $15, 18$, and 10, respectively. Let K be the area of $ABCD$. Find $\sqrt{2} \cdot K$.

Solution:

Since $AB \parallel CD$, the distance 18 from A to CD is the height. Put $D = (0,0), C = (c, 0), A = (a, 18), B = (c-a, 18)$, and let $u = AB = c - 2a > 0$. The point-to-line distance formulas give

$$\frac{18u}{\sqrt{324 + a^2}} = 15 \quad (\text{to } BC),$$
$$\frac{18u}{\sqrt{324 + (u+a)^2}} = 10 \quad (\text{to } BD),$$

so $324 + a^2 = \left(\frac{6u}{5}\right)^2$ and $324 + (u+a)^2 = \left(\frac{9u}{5}\right)^2$.

Subtracting, $u(u+2a) = \frac{81-36}{25}u^2 = \frac{9}{5}u^2$, hence $u + 2a = \frac{9}{5}u$ and $a = \frac{2}{5}u$.
Substituting back, $324 = \frac{36u^2}{25} - \frac{4u^2}{25} = \frac{32u^2}{25}$, so $u^2 = \frac{2025}{8}$ and $u = \frac{45\sqrt{2}}{4}$.

Then $CD = c = u + 2a = \frac{9}{5}u = \frac{81\sqrt{2}}{4}$, and

$$\begin{aligned} K &= \frac{AB + CD}{2} \cdot 18 \\ &= 9\left(\frac{45\sqrt{2}}{4} + \frac{81\sqrt{2}}{4}\right) \\ &= \frac{567\sqrt{2}}{2}, \end{aligned}$$

so $\sqrt{2} \cdot K = 567$.

10. Consider the sequence $(a_k)_{k\geq 1}$ of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n, then

$$a_{k+1} = \frac{m+18}{n+19}.$$

Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\frac{t}{t+1}$ for some positive integer t.

Solution:

Write $a_k = \frac{m}{n}$ in lowest terms and let $d = n - m$, so a_j has the form $\frac{t}{t+1}$ exactly when $d = 1$. One step sends (m, n) to $(m + 18,\ n + 19)$ and then cancels $g = \gcd(m + 18, n + 19)$. Two facts control everything. First, $I = 19m - 18n$ satisfies $19(m + 18) - 18(n + 19) = I$, so I is unchanged by the shift and divided by g upon cancellation. Second, since $I = 19(m + 18) - 18(n + 19) = (m + 18) - 18(d + 1)$, a number divides both $m + 18$ and $n + 19$ exactly when it divides both $d + 1$ and I; hence $g = \gcd(d + 1,\ I)$, and after cancelling, the new difference is $\frac{d+1}{g}$.

Initially $I = 19 \cdot 2020 - 18 \cdot 2021 = 2002 = 2 \cdot 7 \cdot 11 \cdot 13$ and $d = 1$, so $j = 1$ works. The next step has $d + 1 = 2, g = 2 : a_2 = \frac{2038}{2040} = \frac{1019}{1020}$, so $j = 2$ works and $I = 1001 = 7 \cdot 11 \cdot 13$. From there d climbs $1, 2, 3, \ldots$ until $d + 1$ shares a factor with $I :$ at $d + 1 = 7$ we get $a_8 = \frac{161}{162}$ (cancel 7, now $I = 143$); then at $d + 1 = 11, a_{18} = \frac{31}{32}$ (cancel 11, now $I = 13$); then at $d + 1 = 13, a_{30} = \frac{19}{20}$ (cancel 13, now $I = 1$). Each cancellation used $g = d + 1$, so d returned to 1 at $j = 8, 18,$ and 30.

Once $I = 1$, no further cancellation is possible, so d increases forever and never equals 1 again. The valid indices are $j = 1, 2, 8, 18, 30$, with sum $1 + 2 + 8 + 18 + 30 = 59$.

11. Let $ABCD$ be a cyclic quadrilateral with $AB = 4, BC = 5, CD = 6$, and $DA = 7$. Let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to line BD, and let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to line AC. The perimeter of $A_1B_1C_1D_1$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Let $P = AC \cap BD$ and let θ be the acute angle between the diagonals. Since A lies on line AC, its foot A_1 on BD satisfies $PA_1 = PA\cos\theta$, landing on the ray of BD making the acute angle with ray PA; the same holds for all four feet. So $A_1B_1C_1D_1$ is the image of $ABCD$ under the map that rotates each ray from P onto the other diagonal (through angle θ) and scales by $\cos\theta$: corresponding triangles at P are similar with ratio $\cos\theta$, and every side of $A_1B_1C_1D_1$ is $\cos\theta$ times the corresponding side of $ABCD$. Hence the perimeter is $(4+5+6+7)\cos\theta = 22\cos\theta$.

By Ptolemy, $AC \cdot BD = 4 \cdot 6 + 5 \cdot 7 = 59$. By Brahmagupta with $s = 11$, the area is $\sqrt{7 \cdot 6 \cdot 5 \cdot 4} = 2\sqrt{210}$. Since the area also equals $\frac{1}{2}\, AC \cdot BD \sin\theta$, we get $\sin\theta = \frac{4\sqrt{210}}{59}$, so

$$\cos^2\theta = 1 - \frac{3360}{3481} = \frac{121}{3481},$$
$$\cos\theta = \frac{11}{59}.$$

The perimeter is $22 \cdot \frac{11}{59} = \frac{242}{59}$, which is in lowest terms, so $m + n = 242 + 59 = 301.$

12. Let $A_1A_2A_3\ldots A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously, each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Track the three gaps between consecutive frogs around the circle; they start at $(4,4,4)$ and always sum to 12. If the frogs jump by $X_1, X_2, X_3 \in \{\pm 1\}$, the gaps change by $X_2 - X_1, X_3 - X_2, X_1 - X_3$, so each gap stays even and the process stops exactly when some gap becomes 0. Enumerating the 8 equally likely sign choices: from $\{4,4,4\}$, the state stays with probability $\frac{2}{8}$ and moves to $\{2,4,6\}$ with probability $\frac{6}{8}$. From $\{2,4,6\}$: stay with probability $\frac{4}{8}$, move to $\{4,4,4\}$ or $\{2,2,8\}$ with probability $\frac{1}{8}$ each, and stop with probability $\frac{2}{8}$. From $\{2,2,8\}$: stay with probability $\frac{2}{8}$, move to $\{2,4,6\}$ with probability $\frac{2}{8}$, and stop with probability $\frac{4}{8}$.

Let E_1, E_2, E_3 be the expected remaining times from $\{4,4,4\}, \{2,4,6\}, \{2,2,8\}$. Then

$$\begin{aligned} E_1 &= 1 + \tfrac{1}{4}E_1 + \tfrac{3}{4}E_2, \\ E_2 &= 1 + \tfrac{1}{2}E_2 + \tfrac{1}{8}E_1 \\ &\quad + \tfrac{1}{8}E_3, \\ E_3 &= 1 + \tfrac{1}{4}E_3 + \tfrac{1}{4}E_2. \end{aligned}$$

The third gives $E_3 = \frac{4}{3} + \frac{E_2}{3}$; substituting into the second yields $E_2 = 4$, then $E_3 = \frac{8}{3}$ and $E_1 = \frac{4}{3} + E_2 = \frac{16}{3}$.

The expected number of minutes is $\frac{16}{3}$, so $m+n = 16+3 = 19$.

13. Circles ω_1 and ω_2 with radii 961 and $625,$ respectively, intersect at distinct points A and $B.$ A third circle ω is externally tangent to both ω_1 and $\omega_2.$ Suppose line AB intersects ω at two points P and Q such that the measure of minor arc $\widehat{PQ}$ is $120^\circ.$ Find the distance between the centers of ω_1 and $\omega_2.$

Solution:

Let O and r be the center and radius of ω, and O_1, O_2 the other centers. External tangency gives $OO_1 = r + 961,$ so the power of O with respect to ω_1 is $OO_1^2 - 961^2 = r^2 + 2 \cdot 961r;$ similarly its power with respect to ω_2 is $r^2 + 2 \cdot 625r.$ The difference is $2r(961 - 625) = 672r.$

For any point X, the difference $\mathrm{pow}_{\omega_1}(X) - \mathrm{pow}_{\omega_2}(X) = (XO_1^2 - XO_2^2)$ $-(961^2 - 625^2)$ is a linear function of X that vanishes on the radical axis, which is line AB; its rate of change perpendicular to AB is $2 \cdot O_1O_2.$ So the difference equals $2 \cdot O_1O_2 \cdot \mathrm{dist}(O, AB).$ Meanwhile the chord PQ of ω subtends a 120° central angle, so $\mathrm{dist}(O, AB) = r \cos 60^\circ = \frac{r}{2}.$

Therefore $672r = 2 \cdot O_1O_2 \cdot \frac{r}{2} = O_1O_2 \cdot r,$ and the distance between the centers is $672.$

14. For any positive integer $a, \sigma(a)$ denotes the sum of the positive integer divisors of a. Let n be the least positive integer such that $\sigma(a^n) - 1$ is divisible by 2021 for all positive integers a. Find the sum of the prime factors in the prime factorization of n.

Solution:

Note $2021 = 43 \cdot 47$. If $a = \prod p_i^{e_i}$, then $\sigma(a^n) = \prod \sigma(p_i^{e_i n})$, so it suffices (and is necessary, taking a prime) that $\sigma(p^N) \equiv 1$, i.e. $p + p^2 + \cdots + p^N \equiv 0 \pmod{43 \cdot 47}$, for every prime p and every multiple N of n.

Fix $q \in \{43, 47\}$. If $q \mid p$ the sum is 0. If $p \equiv 1 \pmod{q}$ the sum is $\equiv N$, so choosing such a prime (Dirichlet) forces $q \mid n$. Otherwise the sum is $p \cdot \frac{p^N - 1}{p-1}$ with $p - 1$ invertible, so we need $p^N \equiv 1 \pmod{q}$; choosing p to be a primitive root mod q forces $q - 1 \mid n$. Conversely, if $q(q-1) \mid n$ then for every multiple N of n and every prime p, the sum vanishes mod q in all three cases. Hence the least n is

$$\begin{aligned} n &= \operatorname{lcm}(43 \cdot 42,\ 47 \cdot 46) \\ &= 2 \cdot 3 \cdot 7 \cdot 23 \cdot 43 \cdot 47. \end{aligned}$$

The sum of the prime factors is $2 + 3 + 7 + 23 + 43 + 47 = 125$.

15. Let S be the set of positive integers k such that the two parabolas

$$y = x^2 - k$$

and

$$x = 2(y - 20)^2 - k$$

intersect in four distinct points, and these four points lie on a circle with radius at most 21. Find the sum of the least element of S and the greatest element of S.

Solution:

Adding the equation $x^2 - y - k = 0$ to $\frac{1}{2}$ times $2(y-20)^2 - x - k = 0$ gives a conic through all intersection points with equal x^2 and y^2 coefficients:

$$\begin{aligned} x^2 + y^2 - \tfrac{1}{2}x - 41y \\ + 400 - \tfrac{3k}{2} = 0, \end{aligned}$$

a circle centered at $\left(\frac{1}{4}, \frac{41}{2}\right)$ with squared radius $\frac{1}{16} + \frac{1681}{4} - 400 + \frac{3k}{2} = \frac{325}{16} + \frac{3k}{2}$. So whenever four distinct intersection points exist, they are concyclic, and the radius is at most 21 exactly when $\frac{325}{16} + \frac{3k}{2} \leq 441$, i.e. $k \leq 280$ for integers.

Substituting $y = x^2 - k$ into the second parabola gives the quartic $f(x) = 2(x^2 - c)^2 - x - k = 0$ where $c = k + 20$. For $1 \leq k \leq 4$: if $x \leq -k$ then $f(x) = 2(x^2 - c)^2 + (-x - k) > 0$, and if $-k < x \leq 0$ then $x^2 < 16$ while $c \geq 21$, so $f(x) > 2 \cdot 25 - k > 0$; thus there are no intersections with $x \leq 0$, and since $f'(x) = 8x^3 - 8cx - 1$ has exactly one positive root, f has at most (and, by $f(0) > 0$, $f(\sqrt{c}) < 0$, exactly) two positive roots. So $k \leq 4$ fails. For $k \geq 5$, we have $k^2 \geq k + 20$, so $f\left(-\sqrt{c}\right) = \sqrt{c} - k \leq 0$. On $(-\infty, -\sqrt{c}]$, we have $f'(x) = 8x(x^2 - c) - 1 < 0$, so there is exactly one root there (at $-\sqrt{c}$ when $k = 5$). If $k = 5$, the negative derivative at that endpoint makes f negative immediately to its right; if $k > 5$, it is already negative at the endpoint. Since $f(0) = 2c^2 - k > 0$, there is a second root in $(-\sqrt{c}, 0)$. Finally, $f\left(\sqrt{c}\right) = -\sqrt{c} - k < 0$ and $f(+\infty) = +\infty$, so there is one root in each of $(0, \sqrt{c})$ and $(\sqrt{c}, \infty)$. These four roots are distinct, and a quartic has no others.

Hence $S = \{5, 6, \ldots, 280\}$, and the answer is $5 + 280 = 285$.

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