# 2021 AIME II Solutions

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**1.** Find the arithmetic mean of all the three-digit palindromes. (Recall that a palindrome is a number that reads the same forward and backward, such as $777$ or $383$.)

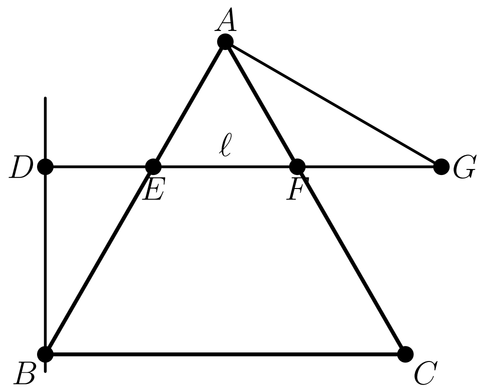
## Solution:

A three-digit palindrome has the form $\overline{aba} = 101a + 10b$ with $a \in \{1, \ldots, 9\}$ and $b \in \{0, \ldots, 9\}$, and every such pair of digits occurs exactly once, so the two digits vary independently over the $90$ palindromes.

By linearity, the mean is $101$ times the average of $a$ plus $10$ times the average of $b$, namely

$$101 \cdot 5 + 10 \cdot \frac{9}{2} = 505 + 45 = 550.$$

**2.** Equilateral triangle $ABC$ has side length $840$. Point $D$ lies on the same side of line $BC$ as $A$ such that $\overline{BD} \perp \overline{BC}$. The line $\ell$ through $D$ parallel to line $BC$ intersects sides $\overline{AB}$ and $\overline{AC}$ at points $E$ and $F$, respectively. Point $G$ lies on $\ell$ such that $F$ is between $E$ and $G$, $\triangle AFG$ is isosceles, and the ratio of the area of $\triangle AFG$ to the area of $\triangle BED$ is $8:9$. Find $AF$.




## Solution:

Since $\ell \parallel BC$, triangle $AEF$ is equilateral; let $s = AF = EF$. The distance between $\ell$ and $BC$ is the height of $ABC$ minus the height of $AEF$, so $BD = \frac{\sqrt{3}}{2}(840 - s)$; write $h = BD$.

In triangle $BED$, the base $\overline{BD}$ is perpendicular to $BC$ and has length $h$, while $E$ lies at horizontal distance $\frac{h}{\sqrt{3}}$ from line $BD$ because $\angle EBC = 60^\circ$. Hence $[BED] = \frac{h^2}{2\sqrt{3}}$. Also $\angle AFG = 180^\circ - \angle AFE = 120^\circ$, and an isosceles triangle with a $120^\circ$ angle must have it as the apex angle, so $FA = FG = s$ and $[AFG] = \frac{1}{2}s^2 \sin 120^\circ = \frac{\sqrt{3}}{4}s^2$.

The ratio condition gives

$$\begin{aligned} \frac{[AFG]}{[BED]} &= \frac{\frac{\sqrt{3}s^2}{4}}{\frac{h^2}{2\sqrt{3}}} \\ &= \frac{3}{2} \cdot \frac{s^2}{h^2} = \frac{8}{9}, \end{aligned}$$

so $\frac{s}{h} = \frac{4}{3\sqrt{3}}$. Substituting $h = \frac{\sqrt{3}}{2}(840 - s)$ yields $s = \frac{2}{3}(840 - s)$, so $5s = 1680$ and $AF = 336$.

**3.** Find the number of permutations $x_1, x_2, x_3, x_4, x_5$ of numbers $1, 2, 3, 4, 5$ such that the sum of five products

$$\begin{aligned} &x_1x_2x_3 + x_2x_3x_4 \\ &\quad + x_3x_4x_5 + x_4x_5x_1 \\ &\quad + x_5x_1x_2 \end{aligned}$$

is divisible by $3$.

## Solution:

Work modulo $3$. The value $3$ is the only multiple of $3$, and each of the five products covers three cyclically consecutive positions, so if $x_i = 3$ exactly two products avoid position $i$ : those covering positions $i+1, i+2, i+3$ and $i+2, i+3, i+4$ (indices mod $5$). Their sum is $x_{i+2}x_{i+3}(x_{i+1} + x_{i+4})$, and since $x_{i+2}x_{i+3}$ is not divisible by $3$, the condition is $x_{i+1} + x_{i+4} \equiv 0 \pmod 3$.

Among the remaining values, $1$ and $4$ are $\equiv 1$ while $2$ and $5$ are $\equiv 2 \pmod 3$, so positions $i+1$ and $i+4$ must take one value from each class: $2 \cdot 2 \cdot 2 = 8$ ordered choices. The other two values fill positions $i+2$ and $i+3$ in $2$ ways. With $5$ choices for the position of $3$, the count is $5 \cdot 8 \cdot 2 = 80$.

**4.** There are real numbers $a, b, c,$ and $d$ such that $-20$ is a root of $x^3 + ax + b$ and $-21$ is a root of $x^3 + cx^2 + d$. These two polynomials share a complex root $m + \sqrt{n} \cdot i$, where $m$ and $n$ are positive integers and $i = \sqrt{-1}$. Find $m + n$.

## Solution:

Both cubics have real coefficients, so their non-real roots come in conjugate pairs: the roots of the first are $-20$ and $m \pm \sqrt{n}\, i$, and the roots of the second are $-21$ and $m \pm \sqrt{n}\, i$.

The first cubic $x^3 + ax + b$ has no $x^2$ term, so its roots sum to $0 : -20 + 2m = 0$, giving $m = 10$. The second cubic $x^3 + cx^2 + d$ has no $x$ term, so the sum of pairwise products of its roots is $0$ :

$$\begin{aligned}&(m + \sqrt{n}\, i)(m - \sqrt{n}\, i) \\ &\quad + (-21)(2m) \\ &= m^2 + n - 42m = 0,\end{aligned}$$

so $n = 420 - 100 = 320$. Then $m + n = 10 + 320 = 330$.

**5.** For positive real numbers $s$, let $\tau(s)$ denote the set of all obtuse triangles that have area $s$ and two sides with lengths $4$ and $10$. The set of all $s$ for which $\tau(s)$ is nonempty, but all triangles in $\tau(s)$ are congruent, is an interval $[a, b)$. Find $a^2 + b^2$.

## Solution:

A triangle with sides $4$ and $10$ is determined by the included angle $\theta$, and its area is $\frac{1}{2} \cdot 4 \cdot 10 \sin\theta = 20 \sin\theta$. When $\theta > 90^\circ$ the triangle is obtuse, and this case produces exactly one triangle for each area $s \in (0, 20)$.

When $\theta < 90^\circ$, the third side satisfies $c^2 = 116 - 80\cos\theta$, and the triangle is obtuse only if the angle opposite the side of length $10$ is obtuse (if $c$ were the longest side, its opposite angle $\theta$ would be acute, making the triangle acute). That requires $4^2 + c^2 < 10^2$, i.e. $116 - 80\cos\theta < 84$, i.e. $\cos\theta > \frac{2}{5}$. Then $\sin\theta < \frac{\sqrt{21}}{5}$, so this second family exists exactly for $s < 4\sqrt{21}$.

For $s < 4\sqrt{21}$ there are two non-congruent obtuse triangles (their third sides differ), while for $4\sqrt{21} \leq s < 20$ only the obtuse-$\theta$ triangle exists: at $s = 4\sqrt{21}$ the acute-$\theta$ candidate becomes a right triangle. For $s \geq 20$ there are none. Hence $[a, b) = [4\sqrt{21}, 20)$ and $a^2 + b^2 = 336 + 400 = 736$.

**6.** For any finite set $S$, let $|S|$ denote the number of elements in $S$. Find the number of ordered pairs $(A, B)$ such that $A$ and $B$ are (not necessarily distinct) subsets of $\{1, 2, 3, 4, 5\}$ that satisfy

$$|A| \cdot |B| = |A \cap B| \cdot |A \cup B|.$$

## Solution:

Let $a = |A|, b = |B|$, and $i = |A \cap B|$, so $|A \cup B| = a + b - i$. The condition $ab = i(a + b - i)$ rearranges to

$$\begin{aligned} &ab - ia - ib + i^2 \\ &= (a - i)(b - i) = 0, \end{aligned}$$

so $|A \cap B| = |A|$ or $|A \cap B| = |B|$. Since $A \cap B$ is a subset of each, that means $A \subseteq B$ or $B \subseteq A$.

For pairs with $A \subseteq B$, each of the $5$ elements independently lies in neither set, in $B$ only, or in both: $3^5 = 243$ pairs. Likewise $243$ pairs satisfy $B \subseteq A$, and the pairs counted twice are exactly those with $A = B$, of which there are $2^5 = 32$. The answer is $243 + 243 - 32 = 454$.

**7.** Let $a, b, c$, and $d$ be real numbers that satisfy the system of equations

$$a + b = -3,$$
$$ab + bc + ca = -4,$$
$$abc + bcd + cda + dab = 14,$$
$$abcd = 30.$$

There exist relatively prime positive integers $m$ and $n$ such that

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}.$$

Find $m + n$.

## Solution:

Since $a + b = -3$, the second equation reads $ab + c(a + b) = ab - 3c = -4$, so $ab = 3c - 4$. Grouping the third equation as $ab(c + d) + cd(a + b) = 14$ gives $(3c - 4)(c + d) - 3cd = 14$, which simplifies to $3c^2 - 4c - 4d = 14$, so $d = \frac{3c^2-4c-14}{4}$. The fourth equation becomes $(3c - 4)\,cd = 30$.
Substituting for $d$ yields $c(3c - 4)(3c^2 - 4c - 14) = 120$, i.e.

$$\begin{aligned} &9c^4 - 24c^3 - 26c^2 \\ &\quad + 56c - 120 \\ &= (c + 2)(3c - 10) \\ &\quad \cdot (3c^2 - 4c + 6) \\ &= 0. \end{aligned}$$

The quadratic factor has negative discriminant, so $c = -2$ or $c = \frac{10}{3}$. If $c = \frac{10}{3}$, then $ab = 6$ with $a + b = -3$, impossible for real $a, b$ since $9 - 24 < 0$. So $c = -2$, giving $ab = -10$ and $d = \frac{12+8-14}{4} = \frac{3}{2}$.
Then $a^2 + b^2 = (a + b)^2 - 2ab = 9 + 20 = 29$ and $c^2 + d^2 = 4 + \frac{9}{4} = \frac{25}{4}$, so $a^2 + b^2 + c^2 + d^2 = \frac{141}{4}$ and $m + n = 141 + 4 = 145$.

**8.** An ant makes a sequence of moves on a cube where a move consists of walking from one vertex to an adjacent vertex along an edge of the cube. Initially the ant is at a vertex of the bottom face of the cube and chooses one of the three adjacent vertices to move to as its first move. For all moves after the first move, the ant does not return to its previous vertex, but chooses to move to one of the other two adjacent vertices. All choices are selected at random so that each of the possible moves is equally likely. The probability that after exactly $8$ moves that ant is at a vertex of the top face on the cube is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

## Solution:

All $3 \cdot 2^7 = 384$ allowed move sequences are equally likely, so we count those ending on the top face. Classify the ant after each move by its face (bottom or top) and by whether its last move was vertical: after a vertical move the two allowed continuations are the two horizontal edges at the new vertex, while after a horizontal move one continuation is horizontal and one is vertical.

Let $(B_h, B_v, T_h, T_v)$ count sequences ending on the bottom or top with last move horizontal or vertical. Each sequence splits into two, following

$$\begin{aligned} B_h' &= B_h + 2B_v, \\ T_v' &= B_h, \\ T_h' &= T_h + 2T_v, \\ B_v' &= T_h. \end{aligned}$$

After move $1$ the counts are $(2, 0, 0, 1)$, and iterating gives $(2, 0, 2, 2)$, $(2, 2, 6, 2)$, $(6, 6, 10, 2)$, $(18, 10, 14, 6)$, $(38, 14, 26, 18)$, $(66, 26, 62, 38)$, and after the eighth move $T_h = 62 + 76 = 138$ and $T_v = 66$.

So $138 + 66 = 204$ of the $384$ sequences end on the top face, giving probability $\frac{204}{384} = \frac{17}{32}$ and $m + n = 17 + 32 = 49$.

**9.** Find the number of ordered pairs $(m, n)$ such that $m$ and $n$ are positive integers in the set $\{1, 2, \ldots, 30\}$ and the greatest common divisor of $2^m + 1$ and $2^n - 1$ is not $1$.

## Solution:

Suppose an odd prime $p$ divides both $2^m + 1$ and $2^n - 1$. From $2^m \equiv -1 \pmod{p}$, the order of $2$ modulo $p$ divides $2m$ but not $m$, so the order contains exactly one more factor of $2$ than $m$ does. The order also divides $n$, so $n$ must contain strictly more factors of $2$ than $m$ : writing $v_2$ for the number of factors of $2$, we need $v_2(n) > v_2(m)$.

Conversely, if $v_2(n) > v_2(m)$, let $g = \gcd(m, n)$. Then $v_2(g) = v_2(m)$, so $\frac{m}{g}$ is odd and $2^g + 1 \mid 2^m + 1$; also $2g \mid n$, so $2^g + 1 \mid 2^{2g} - 1 \mid 2^n - 1$. Hence the gcd exceeds $1$ exactly when $v_2(n) > v_2(m)$.

Among $1, \ldots, 30$ the counts of numbers with $v_2 = 0, 1, 2, 3, 4$ are $15, 8, 4, 2, 1$. The number of pairs with $v_2(m) < v_2(n)$ is

$$\begin{aligned} &15 \cdot 15 + 8 \cdot 7 + 4 \cdot 3 + 2 \cdot 1 \\ &= 225 + 56 + 12 + 2 = 295. \end{aligned}$$

**10.** Two spheres with radii $36$ and one sphere with radius $13$ are each externally tangent to the other two spheres and to two different planes $\mathcal{P}$ and $\mathcal{Q}$. The intersection of planes $\mathcal{P}$ and $\mathcal{Q}$ is the line $\ell$. The distance from line $\ell$ to the point where the sphere with radius $13$ is tangent to plane $\mathcal{P}$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

## Solution:

A sphere of radius $r$ tangent to both planes has its center on the half-plane bisecting the dihedral angle. If the dihedral angle is $2\theta$, the center is at distance $\frac{r}{\sin\theta}$ from $\ell$. In the cross-section through the center perpendicular to $\ell$, the point of $\ell$, the center, and the tangent point on $\mathcal{P}$ form a right triangle with angle $\theta$ at $\ell$, so the tangent point lies at distance $\frac{r\cos\theta}{\sin\theta}$ from $\ell$.

Measure positions along $\ell$. The centers of the two radius-$36$ spheres are both at distance $\frac{36}{\sin\theta}$ from $\ell$ and are $72$ apart, so they differ by $72$ along $\ell$, and by symmetry the radius-$13$ center sits halfway between them along $\ell$, at distance $\frac{13}{\sin\theta}$ from $\ell$. External tangency makes its distance to each big center $49$ :

$$\left(\frac{36}{\sin\theta} - \frac{13}{\sin\theta}\right)^2 + 36^2 = 49^2,$$

so $\left(\frac{23}{\sin\theta}\right)^2 = 49^2 - 36^2 = 1105$. Since $23^2 + 24^2 = 1105$, we get $\sin\theta = \frac{23}{\sqrt{1105}}$ and $\cos\theta = \frac{24}{\sqrt{1105}}$.

The required distance is $\frac{13\cos\theta}{\sin\theta} = \frac{13\cdot 24}{23} = \frac{312}{23}$, which is in lowest terms, so $m + n = 312 + 23 = 335$.

**11.** A teacher was leading a class of four perfectly logical students. The teacher chose a set $S$ of four integers and gave a different number in $S$ to each student. Then the teacher announced to the class that the numbers in $S$ were four consecutive two-digit positive integers, that some number in $S$ was divisible by $6$, and a different number in $S$ was divisible by $7$. The teacher then asked if any of the students could deduce what $S$ is, but in unison, all of the students replied no.

However, upon hearing that all four students replied no, each student was able to determine the elements of $S$. Find the sum of all possible values of the greatest element of $S$.

## Solution:

Call a run any set of four consecutive two-digit integers containing a multiple of $6$ and a different multiple of $7$; the runs are exactly the candidates for $S$ allowed by the announcement. A student holding a number that lies in exactly one run could name $S$ immediately, so the unanimous “no” reveals that every element of $S$ lies in at least two runs.

A number belongs to two different runs only when nearby runs overlap, which happens when a multiple of $6$ and a multiple of $7$ are consecutive integers, both two-digit: the pairs $(35, 36)$, $(48, 49)$, $(77, 78)$, and $(90, 91)$. Checking each cluster, the runs all four of whose elements are ambiguous are exactly the ones with such a pair in the two middle positions:

$$\begin{gathered}\{34, 35, 36, 37\},\\ \{47, 48, 49, 50\},\\ \{76, 77, 78, 79\},\\ \{89, 90, 91, 92\}.\end{gathered}$$

These four sets are pairwise disjoint, so after the four “no” replies each student’s own number singles out one of them, consistent with everyone then deducing $S$.

The possible greatest elements are $37, 50, 79,$ and $92,$ with sum $258.$

**12.** A convex quadrilateral has area $30$ and side lengths $5, 6, 9,$ and $7,$ in that order. Denote by $\theta$ the measure of the acute angle formed by the diagonals of the quadrilateral. Then $\tan\theta$ can be written in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

## Solution:

Label the quadrilateral $ABCD$ with $AB=5, BC=6, CD=9, DA=7,$ and let the diagonals meet at $P$, cutting $\overline{AC}$ into $p_1, p_2$ and $\overline{BD}$ into $q_1, q_2$. With $\varphi = \angle APB$, the law of cosines in the four corner triangles (whose angles at $P$ alternate between $\varphi$ and $180^\circ - \varphi$) gives

$$\begin{aligned}&BC^2+DA^2-AB^2-CD^2\\&=2(p_1q_1+p_1q_2+p_2q_1+p_2q_2)\\&\quad\cdot\cos\varphi\\&=2\,AC\cdot BD\cos\varphi.\end{aligned}$$

The left side is $36+49-25-81=-21$, so $AC\cdot BD\,|\cos\varphi|=\frac{21}{2}$, and the acute angle $\theta$ between the diagonals satisfies $AC\cdot BD\cos\theta=\frac{21}{2}$. Meanwhile the four corner triangles give the area $\frac{1}{2}AC\cdot BD\sin\theta=30$, so $AC\cdot BD\sin\theta=60.$ Dividing, $\tan\theta=\frac{60}{\frac{21}{2}}=\frac{40}{7}$, so $m+n=40+7=47.$

**13.** Find the least positive integer $n$ for which $2^n + 5^n - n$ is a multiple of $1000$.

## Solution:

Work modulo $8$ and $125$. For $n \geq 3$ we have $2^n \equiv 0 \pmod 8$, so we need $n \equiv 5^n \pmod 8$. If $n$ is even then $5^n \equiv 1$, forcing the even number $n$ to be $\equiv 1 \pmod 8$, impossible; so $n$ is odd, $5^n \equiv 5$, and $n \equiv 5 \pmod 8$. Also $5^n \equiv 0 \pmod{125}$ for $n \geq 3$, so we need $n \equiv 2^n \pmod{125}$.

The order of $2$ is $4$ modulo $5$, $20$ modulo $25$, and $100$ modulo $125$. Since $n \equiv 5 \pmod 8$ gives $n \equiv 1 \pmod 4$, we get $2^n \equiv 2 \pmod 5$, so $n \equiv 2 \pmod 5$ and hence $n \equiv 17 \pmod{20}$. Then $2^n \equiv 2^{17} = 2^{10} \cdot 2^7 \equiv (-1)(3) \equiv 22 \pmod{25}$, so $n \equiv 22 \pmod{25}$, which with $n \equiv 1 \pmod 4$ gives $n \equiv 97 \pmod{100}$. Finally $2^{10} \equiv 24, 2^{20} \equiv 76, 2^{40} \equiv 26, 2^{80} \equiv 51 \pmod{125}$, so

$$\begin{aligned} 2^n &\equiv 2^{97} = 2^{80} \cdot 2^{10} \cdot 2^7 \\ &\equiv 51 \cdot 24 \cdot 3 \equiv 47 \pmod{125}, \end{aligned}$$

giving $n \equiv 47 \pmod{125}$.

Combining $n \equiv 47 \pmod{125}$ with $n \equiv 5 \pmod 8$ yields $n \equiv 797 \pmod{1000}$, and $n = 1, 2$ fail by direct check, so the least such $n$ is $797$.

**14.** Let $\triangle ABC$ be an acute triangle with circumcenter $O$ and centroid $G$. Let $X$ be the intersection of the line tangent to the circumcircle of $\triangle ABC$ at $A$ and the line perpendicular to $GO$ at $G$. Let $Y$ be the intersection of lines $XG$ and $BC$. Given that the measures of $\angle ABC$, $\angle BCA$, and $\angle XOY$ are in the ratio $13 : 2 : 17$, the degree measure of $\angle BAC$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

## Solution:

Let $M$ be the midpoint of $\overline{BC}$, so $A, G, M$ are collinear along the median, while $X$, $G$, $Y$ are collinear by definition. Since $OA \perp AX$ (tangent and radius) and $OG \perp GX$, quadrilateral $OAXG$ is cyclic with diameter $\overline{OX}$. Since $OG \perp GY$ and $OM \perp MY$ (the segment from the center to the midpoint of a chord is perpendicular to it), quadrilateral $OGYM$ is cyclic with diameter $\overline{OY}$.

In each circle the chord $\overline{OG}$ subtends equal angles, so $\angle OXY = \angle OXG = \angle OAG = \angle OAM$ and $\angle OYX = \angle OYG = \angle OMG = \angle OMA$. Triangles $OXY$ and $OAM$ therefore have the same angle sums at their bases, giving

$$\begin{aligned}\angle XOY &= 180^\circ - \angle OXY \\ &\quad - \angle OYX \\ &= 180^\circ - \angle OAM \\ &\quad - \angle OMA \\ &= \angle AOM.\end{aligned}$$

Write $\angle ABC = 13k$ and $\angle BCA = 2k$, so $\angle BAC = 180^\circ - 15k$. Central angles give $\angle AOB = 2\angle BCA = 4k$, and $\overline{OM}$ bisects $\angle BOC = 2\angle BAC$, so on the side of $B$ (nearer to $A$'s arc since $\angle ABC > \angle BCA$),

$$\begin{aligned}\angle AOM &= \angle AOB + \angle BOM \\ &= 4k + (180^\circ - 15k) \\ &= 180^\circ - 11k.\end{aligned}$$

Setting $180^\circ - 11k = \angle XOY = 17k$ gives $k = \frac{45}{7}$, so $\angle BAC = 180^\circ - 15 \cdot \frac{45}{7} =$

$\frac{585}{7}$ degrees, and all three angles are acute as required. Then $m + n = 585 + 7 = 592.$

**15.** Let $f(n)$ and $g(n)$ be functions satisfying

$$f(n) = \begin{cases} \sqrt{n} & \text{if } \sqrt{n} \text{ is an integer} \\ 1 + f(n+1) & \text{otherwise} \end{cases}$$

and

$$g(n) = \begin{cases} \sqrt{n} & \text{if } \sqrt{n} \text{ is an integer} \\ 2 + g(n+2) & \text{otherwise} \end{cases}$$

for positive integers $n$. Find the least positive integer $n$ such that $\frac{f(n)}{g(n)} = \frac{4}{7}$.

## Solution:

Let $k$ be the least integer with $k^2 \geq n$. The function $f$ climbs one step at a time to the next perfect square, so $f(n) = k + (k^2 - n)$. The function $g$ climbs by $2$s, preserving the parity of its argument, and $j^2 \equiv j \pmod 2$, so $g(n) = j + (j^2 - n)$ where $j$ is the least integer with $j^2 \geq n$ and $j \equiv n \pmod 2$. If $k \equiv n \pmod 2$ then $j = k$ and the ratio is $1$; so we need $k \not\equiv n \pmod 2$, in which case $j = k + 1$ and $g(n) - f(n) = (k+1)^2 - k^2 + 1 = 2k + 2$.

Then $7f = 4g = 4f + 4(2k+2)$ gives $f = \frac{8(k+1)}{3}$, so $k \equiv 2 \pmod 3$ and

$$n = k^2 + k - \frac{8(k+1)}{3},$$

subject to $(k-1)^2 < n \leq k^2$ and $n \not\equiv k \pmod 2$.

For $k = 2, 5, 8, 11$ the formula gives $n = -2, 14, 48, 100$, each failing $n > (k-1)^2$; for $k = 14, n = 170$ is in range but has the same parity as $k$. For $k = 17, n = 289 + 17 - 48 = 258$, which satisfies $256 < 258 \leq 289$ and is even while $k$ is odd. Indeed $f(258) = 17 + 31 = 48$ and $g(258) = 18 + 66 = 84$, with $\frac{48}{84} = \frac{4}{7}$, so the least $n$ is $258$.

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