2004 AMC 10B Solutions

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1. Each row of the Misty Moon Amphitheater has 33 seats. Rows 12 through 22 are reserved for a youth club. How many seats are reserved for this club?

A 297

B 330

C 363

D 396

E 726

Solution:

Rows 12 through 22 inclusive make up $22 - 12 + 1 = 11$ rows.

Each row has 33 seats, so the total is $33 \times 11 = 363.$

Thus, the correct answer is **C**.

2. How many two-digit positive integers have at least one 7 as a digit?

A 10

B 18

C 19

D 20

E 30

Solution:

The numbers 70 through 79 give 10 with a 7 in the tens place.

The numbers $17, 27, \ldots, 97$ give 9 with a 7 in the units place.

Since 77 is counted twice, the total is $10 + 9 - 1 = 18.$

Thus, the correct answer is **B**.

3. At each basketball practice last week, Jenny made twice as many free throws as she made at the previous practice. At her fifth practice she made 48 free throws. How many free throws did she make at the first practice?

A 3

B 6

C 9

D 12

E 15

Solution:

Working backward from the fifth practice, the counts are $48, 24, 12, 6,$ and 3 at the fourth, third, second, and first practices.

Thus, the correct answer is **A**.

4. A standard six-sided die is rolled, and P is the product of the five numbers that are visible. What is the largest number that is certain to divide P?

A 6

B 12

C 24

D 144

E 720

Solution:

Since $6! = 720 = 2^4 \cdot 3^2 \cdot 5,$ the visible product uses only the primes $2, 3,$ and $5.$

Hiding 4 leaves the fewest 2's, namely $2^2.$ Hiding 3 or 6 leaves the fewest 3's, namely one. Hiding 5 leaves no factor of $5.$

Therefore P is always divisible by $2^2 \cdot 3 = 12,$ but not necessarily by any larger number.

Thus, the correct answer is **B**.

5. In the expression $c \cdot a^b - d$, the values of $a, b, c,$ and d are $0, 1, 2,$ and $3,$ although not necessarily in that order. What is the maximum possible value of the result?

- A 5
- B 6
- C 8
- D 9
- E 10

Solution:

Setting $d = 0$ removes the subtraction, so we maximize $c \cdot a^b$ using $1, 2, 3.$

Taking $c = 1, a = 3, b = 2$ gives $3^2 = 9.$ The alternative $2^3 = 8$ is smaller, and any assignment with $c > 1$ forces a smaller power. The maximum is $9.$

Thus, the correct answer is **D**.

6. Which of the following numbers is a perfect square?

A $98! \cdot 99!$

B $98! \cdot 100!$

C $99! \cdot 100!$

D $99! \cdot 101!$

E $100! \cdot 101!$

Solution:

For $m < n$, we have $m! \cdot n! = (m!)^2 \cdot (m+1)(m+2)\cdots n$, which is a perfect square precisely when $(m+1)\cdots n$ is a perfect square.

For the five choices this leftover factor is $99, 99 \cdot 100, 100, 100 \cdot 101,$ and $101.$ Only $100 = 10^2$ is a perfect square.

Therefore $99! \cdot 100! = (99! \cdot 10)^2$ is the perfect square.

Thus, the correct answer is **C**.

7. On a trip from the United States to Canada, Isabella took d U.S. dollars. At the border she exchanged them all, receiving 10 Canadian dollars for every 7 U.S. dollars. After spending 60 Canadian dollars, she had d Canadian dollars left. What is the sum of the digits of d?

A 5

B 6

C 7

D 8

E 9

Solution:

Isabella received $\frac{10}{7}d$ Canadian dollars and spent $60,$ leaving $d.$ So

$$\frac{10}{7}d - 60 = d.$$

Then $\frac{3}{7}d = 60,$ so $d = 140.$ The sum of its digits is $1 + 4 + 0 = 5.$

Thus, the correct answer is **A**.

8. Minneapolis-St. Paul International Airport is 8 miles southwest of downtown St. Paul and 10 miles southeast of downtown Minneapolis. Which of the following is closest to the number of miles between downtown St. Paul and downtown Minneapolis?

- A 13
- B 14
- C 15
- D 16
- E 17

Solution:

The two given directions are perpendicular, so the airport sits at the right angle of a right triangle with legs 8 and $10.$

The distance between the downtowns is $\sqrt{8^2 + 10^2} = \sqrt{164} \approx 12.8,$ which is closest to $13.$

Thus, the correct answer is **A**.

9. A square has sides of length 10, and a circle centered at one of its vertices has radius 10. What is the area of the union of the regions enclosed by the square and the circle?

- A $200 + 25\pi$
- B $100 + 75\pi$
- C $75 + 100\pi$
- D $100 + 100\pi$
- E $100 + 125\pi$

Solution:

The square has area $10^2 = 100$ and the circle has area $\pi(10)^2 = 100\pi$.

Since the circle is centered at a vertex of the square, exactly one quarter of the circle, area 25π, lies inside the square.

The union has area $100 + 100\pi - 25\pi = 100 + 75\pi$.

Thus, the correct answer is **B**.

10. A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

- A 5
- B 8
- C 9
- D 10
- E 11

Solution:

The rows hold $1, 3, 5, \ldots$ cans, and the sum of the first n odd numbers is n^2.

Setting $n^2 = 100$ gives $n = 10$.

Thus, the correct answer is **D**.

11. Two eight-sided dice each have faces numbered 1 through 8. When the dice are rolled, each face has an equal probability of appearing on the top. What is the probability that the product of the two top numbers is greater than their sum?

A $\frac{1}{2}$

B $\frac{47}{64}$

C $\frac{3}{4}$

D $\frac{55}{64}$

E $\frac{7}{8}$

Solution:

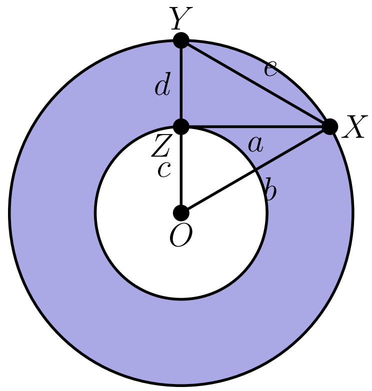
There are $8 \cdot 8 = 64$ ordered pairs. The inequality $mn > m + n$ is equivalent to $(m-1)(n-1) > 1$.

This fails only when $m = 1, n = 1$, or $m = n = 2$, which account for $8 + 8 - 1 + 1 = 16$ pairs.

The probability is $\frac{64 - 16}{64} = \frac{48}{64} = \frac{3}{4}$.

Thus, the correct answer is **C**.

12. An *annulus* is the region between two concentric circles. The concentric circles in the figure have radii b and c, with $b > c$. Let $\overline{OX}$ be a radius of the larger circle, let $\overline{XZ}$ be tangent to the smaller circle at Z, and let $\overline{OY}$ be the radius of the larger circle that contains Z. Let $a = XZ, d = YZ$, and $e = XY$. What is the area of the annulus?



A πa^2

B πb^2

C πc^2

D πd^2

E πe^2

Solution:

The annulus is the difference of the two circular areas, $\pi b^2 - \pi c^2$.

Because $\overline{XZ}$ is tangent to the small circle at Z, it is perpendicular to the radius $\overline{OZ}$. In right triangle OZX with $OX = b, OZ = c$, and $XZ = a$, we get $b^2 - c^2 = a^2$.

Therefore the area of the annulus is πa^2.

Thus, the correct answer is **A**.

13. In the United States, coins have the following thicknesses: penny, 1.55 mm; nickel, 1.95 mm; dime, 1.35 mm; quarter, 1.75 mm. If a stack of these coins is exactly 14 mm high, how many coins are in the stack?

A 7

B 8

C 9

D 10

E 11

Solution:

Measure every thickness in hundredths of a millimeter. The four possible thicknesses are $135, 155, 175, 195,$ all congruent to $15 \pmod{20}$. If the stack has k coins and an integer height, its total in hundredths is divisible by $20,$ so $15k$ is divisible by $20.$ Therefore k must be a multiple of $4.$

A stack of 4 coins is at most $4(1.95) = 7.8$ mm, and a stack of 12 coins is at least $12(1.35) = 16.2$ mm, so only 8 coins can total 14 mm.

Indeed, 8 quarters give $8(1.75) = 14$ mm.

Thus, the correct answer is **B**.

14. A bag initially contains red marbles and blue marbles only, with more blue than red. Red marbles are added to the bag until only $\frac{1}{3}$ of the marbles in the bag are blue. Then yellow marbles are added to the bag until only $\frac{1}{5}$ of the marbles in the bag are blue. Finally, the number of blue marbles in the bag is doubled. What fraction of the marbles now in the bag are blue?

A $\frac{1}{5}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{2}{5}$

E $\frac{1}{2}$

Solution:

Let there be B blue marbles. After adding red marbles the total is $3B$; after adding yellow marbles the total is $5B$, still with B blue.

Doubling the blue marbles gives $2B$ blue out of $6B$ total, which is $\frac{2B}{6B} = \frac{1}{3}$.

Thus, the correct answer is **C**.

15. Patty has 20 coins consisting of nickels and dimes. If her nickels were dimes and her dimes were nickels, she would have 70 cents more. How much are her coins worth?

- A $\$1.15$
- B $\$1.20$
- C $\$1.25$
- D $\$1.30$
- E $\$1.35$

Solution:

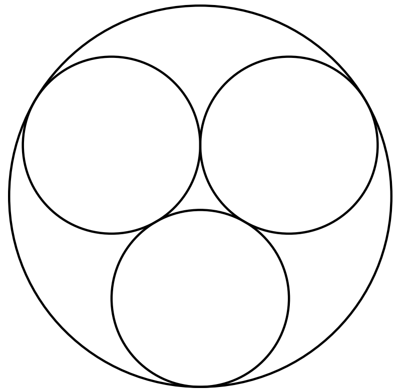
Swapping increases the value, so Patty has more nickels than dimes. Each swapped coin changes the total by 5 cents, so she has $\frac{70}{5} = 14$ more nickels than dimes.

With $n + d = 20$ and $n - d = 14,$ she has 17 nickels and 3 dimes.

Her coins are worth $17 \cdot 5 + 3 \cdot 10 = 115$ cents, or $\$1.15.$

Thus, the correct answer is **A**.

16. Three circles of radius 1 are externally tangent to each other and internally tangent to a larger circle. What is the radius of the large circle?



A $\dfrac{2+\sqrt{6}}{3}$

B 2

C $\dfrac{2+3\sqrt{2}}{3}$

D $\dfrac{3+2\sqrt{3}}{3}$

E $\dfrac{3+\sqrt{3}}{2}$

Solution:

The centers of the three unit circles form an equilateral triangle with side 2. Its center is the center of the large circle.

The distance from the center of an equilateral triangle to a vertex is $\dfrac{\text{side}}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$.

Adding the unit radius, the large radius is $1 + \dfrac{2\sqrt{3}}{3} = \dfrac{3+2\sqrt{3}}{3}$.

Thus, the correct answer is **D**.

17. The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years Jack will be twice as old as Bill will be then. What is the difference in their current ages?

- A 9
- B 18
- C 27
- D 36
- E 45

Solution:

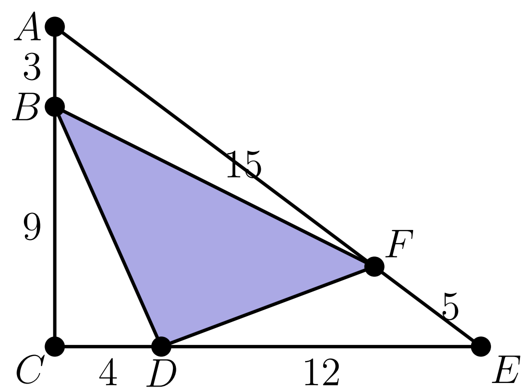
Let Jack's age be $10x + y$ and Bill's be $10y + x$. In five years $10x + y + 5 = 2(10y + x + 5)$, which simplifies to $8x = 19y + 5$.

Since x and y are digits, the only solution is $y = 1, x = 3$.

So Jack is 31 and Bill is 13, a difference of 18.

Thus, the correct answer is **B**.

18. In right triangle $\triangle ACE$, we have $AC = 12, CE = 16$, and $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3, CD = 4$, and $EF = 5$. What is the ratio of the area of $\triangle BDF$ to that of $\triangle ACE$?



A $\frac{1}{4}$

B $\frac{9}{25}$

C $\frac{3}{8}$

D $\frac{11}{25}$

E $\frac{7}{16}$

Solution:

The area of $\triangle ACE$ is $\frac{1}{2}(12)(16) = 96$.

Each corner triangle $\triangle ABF, \triangle BCD$, and $\triangle DEF$ has a base and an altitude that are $\frac{3}{4}$ and $\frac{1}{4}$ of a corresponding base and altitude of $\triangle ACE$. So each has area $\frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$ of $\triangle ACE$.

Hence

$$\frac{[BDF]}{[ACE]} = 1 - 3 \cdot \frac{3}{16}$$
$$= 1 - \frac{9}{16} = \frac{7}{16}.$$

Thus, the correct answer is **E**.

19. In the sequence $2001, 2002, 2003, \ldots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004th term in this sequence?

A -2004

B -2

C 0

D 4003

E 6007

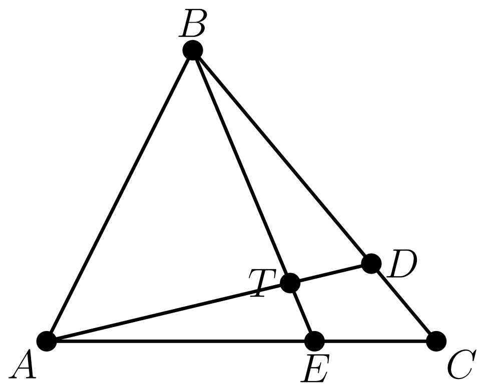
Solution:

The recurrence $a_{k+1} = a_{k-2} + a_{k-1} - a_k$ gives $a_{k+1} - a_{k-1} = -(a_k - a_{k-2})$. The sequence begins $2001, 2002, 2003, 2000, 2005, 1998, \ldots$

So the even-position terms form the arithmetic sequence $2002, 2000, 1998, \ldots$ with common difference -2. The 2004th term is its 1002nd term, $2002 + 1001(-2) = 0$.

Thus, the correct answer is **C**.

20. In $\triangle ABC$ points D and E lie on $\overline{BC}$ and $\overline{AC}$, respectively. If $\overline{AD}$ and $\overline{BE}$ intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?



A $\frac{1}{8}$

B $\frac{2}{9}$

C $\frac{3}{10}$

D $\frac{4}{11}$

E $\frac{5}{12}$

Solution:

Let F be on $\overline{AC}$ with $DF \parallel BE$, and write $ET = x, BT = 4x$.

From $\triangle ATE \sim \triangle ADF$, $\dfrac{DF}{x} = \dfrac{AD}{AT} = \dfrac{4}{3}$, so $DF = \dfrac{4x}{3}$.

From $\triangle BEC \sim \triangle DFC$, $\dfrac{CD}{BC} = \dfrac{DF}{BE} = \dfrac{\frac{4x}{3}}{5x} = \dfrac{4}{15}$.

Therefore

$$\frac{CD}{BD} = \frac{\frac{CD}{BC}}{1 - \frac{CD}{BC}}$$
$$= \frac{\frac{4}{15}}{\frac{11}{15}} = \frac{4}{11}.$$

Thus, the correct answer is **D**.

21. Let $1, 4, \ldots$ and $9, 16, \ldots$ be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

A 3722

B 3732

C 3914

D 3924

E 4007

Solution:

The first sequence is $1 + 3k$ with largest term 6010, and the second is $9 + 7j$ with a much larger last term, so the binding limit is 6010.

A common value has the form $16 + 21m$ (the first shared term is 16, spaced by $\operatorname{lcm}(3, 7) = 21$). Requiring $16 + 21m \leq 6010$ gives $0 \leq m \leq 285$, that is 286 common numbers.

The number of distinct values is $2004 + 2004 - 286 = 3722$.

Thus, the correct answer is **A**.

22. A triangle with sides of $5, 12,$ and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

A $\frac{3\sqrt{5}}{2}$

B $\frac{7}{2}$

C $\sqrt{15}$

D $\frac{\sqrt{65}}{2}$

E $\frac{9}{2}$

Solution:

Since $5^2 + 12^2 = 13^2$, the triangle is right. Place it at $(0,0), (5,0), (0,12)$. The circumcenter is the midpoint of the hypotenuse, $\left(\frac{5}{2}, 6\right)$.

The inradius satisfies $(12-r)+(5-r)=13$, so $r=2$ and the incenter is $(2,2)$.

The distance is

$$\sqrt{\left(\frac{5}{2}-2\right)^2 + (6-2)^2}$$
$$= \sqrt{\frac{1}{4}+16}$$
$$= \frac{\sqrt{65}}{2}.$$

Thus, the correct answer is **D**.

23. Each face of a cube is painted either red or blue, each with probability $\frac{1}{2}$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

A $\frac{1}{4}$

B $\frac{5}{16}$

C $\frac{3}{8}$

D $\frac{7}{16}$

E $\frac{1}{2}$

Solution:

Fixing the orientation, there are $2^6 = 64$ colorings.

A coloring works if all six faces match (2 ways), exactly five match ($\binom{6}{5} \cdot 2 = 12$ ways), or four faces share a color with the remaining pair being opposite faces of the other color (3 opposite pairs, 2 colors, giving 6 ways).

The total is $2 + 12 + 6 = 20,$ so the probability is $\dfrac{20}{64} = \dfrac{5}{16}$.

Thus, the correct answer is **B**.

24. In $\triangle ABC$ we have $AB = 7$, $AC = 8$, and $BC = 9$. Point D is on the circumscribed circle of the triangle so that $\overline{AD}$ bisects $\angle BAC$. What is the value of $\frac{AD}{CD}$?

A $\frac{9}{8}$

B $\frac{5}{3}$

C 2

D $\frac{17}{7}$

E $\frac{5}{2}$

Solution:

Let $\overline{AD}$ meet $\overline{BC}$ at E. Since $\angle ABC$ and $\angle ADC$ subtend the same arc, they are equal, and $\angle EAB = \angle CAD$, so $\triangle ABE \sim \triangle ADC$.

Hence $\dfrac{AD}{CD} = \dfrac{AB}{BE}$.

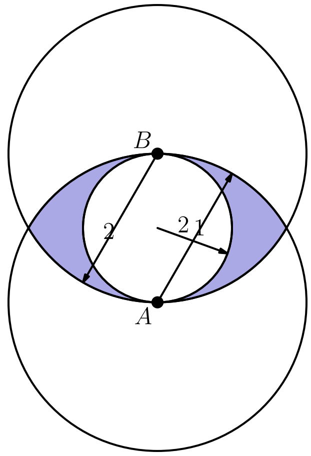
By the Angle Bisector Theorem, $\dfrac{BE}{EC} = \dfrac{AB}{AC}$, so $BE = \dfrac{AB \cdot BC}{AB + AC} = \dfrac{7 \cdot 9}{15}$.

Therefore

$$\begin{aligned}\frac{AD}{CD} &= \frac{AB}{BE} = \frac{AB + AC}{BC} \\ &= \frac{15}{9} = \frac{5}{3}.\end{aligned}$$

Thus, the correct answer is **B**.

25. A circle of radius 1 is internally tangent to two circles of radius 2 at points A and B, where AB is a diameter of the smaller circle. What is the area of the region, shaded in the figure, that is outside the smaller circle and inside each of the two larger circles?



A $\frac{5}{3}\pi - 3\sqrt{2}$

B $\frac{5}{3}\pi - 2\sqrt{3}$

C $\frac{8}{3}\pi - 3\sqrt{3}$

D $\frac{8}{3}\pi - 3\sqrt{2}$

E $\frac{8}{3}\pi - 2\sqrt{3}$

Solution:

Let the large circles have centers A and B, let C be the center of the small circle, and let D be a point where the two large circles meet.

Then $\triangle ACD$ is right with $AC = 1$ and $AD = 2$, so $CD = \sqrt{3}, \angle CAD = 60^\circ$, and its area is $\frac{\sqrt{3}}{2}$.

One quarter of the shaded region equals the 60° sector of the radius-2 circle (area $\frac{2\pi}{3}$) minus $\triangle ACD$ (area $\frac{\sqrt{3}}{2}$) minus a quarter of the small circle (area $\frac{\pi}{4}$), giving $\frac{2\pi}{3} -$

$$\frac{\sqrt{3}}{2} - \frac{\pi}{4} = \frac{5\pi}{12} - \frac{\sqrt{3}}{2}.$$

Multiplying by 4, the shaded area is $\frac{5\pi}{3} - 2\sqrt{3}.$

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc10/2004B