2013 AMC 10B Solutions

Typeset by: LIVE, by Po-Shen Loh
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1. What is

$$\frac{2+4+6}{1+3+5}-\frac{1+3+5}{2+4+6}?$$

A -1

B $\frac{5}{36}$

C $\frac{7}{12}$

D $\frac{49}{20}$

E $\frac{43}{3}$

Solution(s):

$$\begin{aligned}&\frac{2+4+6}{1+3+5}-\frac{1+3+5}{2+4+6}\\&=\frac{12}{9}-\frac{9}{12}\\&=\frac{4}{3}-\frac{3}{4}\\&=\frac{7}{12}\end{aligned}$$

Thus, the correct answer is **C**.

2. Mr. Green measures his rectangular garden by walking two of the sides and finds that it is 15 steps by 20 steps. Each of Mr. Green's steps is 2 feet long. Mr. Green expects a half a pound of potatoes per square foot from his garden. How many pounds of potatoes does Mr. Green expect from his garden?

A 600

B 800

C 1000

D 1200

E 1400

Solution(s):

The dimensions of the garden are $2 \cdot 15 = 30$ feet by $2 \cdot 20 = 40$ feet. Thus, the square footage is $30 \cdot 40 = 1200$.

Therefore, there are

$$1200 \cdot \frac{1}{2} = 600$$

pounds of potatoes.

Thus, the correct answer is **A**.

3. On a particular January day, the high temperature in Lincoln, Nebraska, was 16 degrees higher than the low temperature, and the average of the high and low temperatures was 3 degrees. What was the low temperature in Lincoln that day (in degrees)?

- A -13
- B -8
- C -5
- D -3
- E 11

Solution(s):

Let the low temperature be represented by l. Then, the high temperature is $l + 16$.

This makes the average

$$\frac{l+l+16}{2} = l+8 = 3.$$

Therefore, $l = -5$.

Thus, the correct answer is **C**.

4. When counting from 3 to 201, 53 is the 51^{st} number counted. When counting backwards from 201 to 3, 53 is the n^{th} number counted. What is n?

- A 146
- B 147
- C 148
- D 149
- E 150

Solution(s):

When counting from 3 to 201 in any direction, we count 199 numbers. When counting in the reverse direction, after counting 53, there are 50 more count, meaning 149 were counted by that point.

Thus, the correct answer is **D**.

5. Positive integers a and b are each less than 6. What is the smallest possible value for

$$2 \cdot a - a \cdot b?$$

A -20

B -15

C -10

D 0

E 2

Solution(s):

This expressionS is equal to $a(2-b)$. Thus, we can minimize the value of the expression by making it as negative as possible. This happens when a and b are are large as possible, which is when $a = b = 5$. This makes the expression evaluate to

$$(2-5) \cdot 5 = -15.$$

Thus, the correct answer is **B**.

6. The average age of 33 fifth-graders is 11. The average age of 55 of their parents is 33. What is the average age of all of these parents and fifth-graders?

- A 22
- B 23.25
- C 24.75
- D 26.25
- E 28

Solution(s):

The sum of the ages of all the fifth-graders and their parents is:

$$33 \cdot 11 + 55 \cdot 33 = 66 \cdot 33.$$

Then, as the average is the sum divided by the number of people, the average age must be:

$$\frac{66 \cdot 33}{88} = \frac{3}{4} \cdot 33 = 24.75.$$

Thus, the correct answer is **C**.

7. Six points are equally spaced around a circle of radius 1. Three of these points are the vertices of a triangle that is neither equilateral nor isosceles. What is the area of this triangle?

A $\frac{\sqrt{3}}{3}$

B $\frac{\sqrt{3}}{2}$

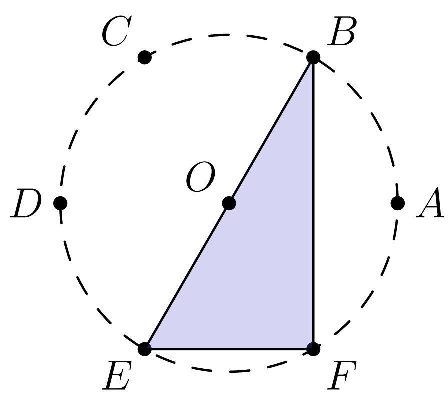
C 1

D $\sqrt{2}$

E 2

Solution(s):

Consider the following diagram:



Working with the above diagram, observe that $\triangle BEF$ is a right triangle.

Furthermore, as each point is equally interspaced across the unit circle,

$$\angle XOB = \frac{2\pi}{6} = \frac{\pi}{3}$$

Therefore, by similarity, $\angle E = \frac{\pi}{3}$. As such, $\triangle BEF$ is a $30-60-90$ triangle. This means that it is not an equilateral triangle nor an isosceles triangle.

Now, as the radius of the circle is 1, $\overline{BE} = 2$.

Thus, by our special right triangle formulas (or just plain triginometry), we have:

$$\overline{EF} = 1$$

$$\overline{BF} = \sqrt{3}$$

As such, the area is equal to $\frac{\sqrt{3}}{2}$

Thus, the correct answer is **B**.

8. Ray's car averages 40 miles per gallon of gasoline, and Tom's car averages 10 miles per gallon of gasoline. Ray and Tom each drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline?

- A 10
- B 16
- C 25
- D 30
- E 40

Solution(s):

When they drive 40 miles each, they in total drive 80 miles. Then, Ray uses 1 gallons and Tom uses 4 gallons, so they used 5 gallons combinded. This makes the milage $\frac{80}{5} = 16$.

Thus, the correct answer is **B**.

9. Three positive integers are each greater than 1, have a product of 27000, and are pairwise relatively prime. What is their sum?

A 100

B 137

C 156

D 160

E 165

Solution(s):

Only one of them is a multiple of 2, only one of them is a multiple of 3, and only one of them is a multiple of 5.

Since each of the positive integers is greater than 1, each of them must be a multiple of one of the given primes.

Therefore, since

$$27000 = 2^3 \cdot 3^3 \cdot 5^3,$$

the numbers must be $2^3, 3^3, 5^3$, making their sum 160.

Thus, the correct answer is **D**.

10. A basketball team's players were successful on 50% of their two-point shots and 40% of their three-point shots, which resulted in 54 points. They attempted 50% more two-point shots than three-point shots. How many three-point shots did they attempt?

A 10

B 15

C 20

D 25

E 30

Solution(s):

Let the number of three-point attempts be t.

Then, the number of made three-point shots is $0.4t$, implying that the number of points made off of three-point shots is $1.2t$.

Similarly, the number of two-point shots is $1.5t$, so the number of made two-point shots is $0.75t$, suggesting that the number of points made off of 2 point shots is $1.5t$.

This makes the total number of points scored equal to:

$$2.7t = 54,$$

and so, $t = 20$.

Thus, the correct answer is **C**.

11. Real numbers x and y satisfy the equation

$$x^2 + y^2 = 10x - 6y - 34.$$

What is $x + y$?

- A 1
- B 2
- C 3
- D 6
- E 8

Solution(s):

This can be rewritten as

$$x^2 - 10x + y^2 + 6y + 34 = 0.$$

From this, completing the square yields

$$x^2 - 10x + 25 + y^2 + 6y + 9$$

$$= (x-5)^2 + (y+3)^2$$

$$= 0.$$

Since both of these squared terms must be greater than or equal to 0, and their sum equals 0, both values must both be 0 yielding

$$(x-5)^2 = (y+3)^2 = 0.$$

As such, $x = 5, y = -3$. This makes the sum $x + y = 2$.

Thus, the correct answer is **B**.

12. Let S be the set of sides and diagonals of a regular pentagon. A pair of elements of S are selected at random without replacement. What is the probability that the two chosen segments have the same length?

A $\frac{2}{5}$

B $\frac{4}{9}$

C $\frac{1}{2}$

D $\frac{5}{9}$

E $\frac{4}{5}$

Solution(s):

We can solve for the number of sides and diagonals as each edge in question is made up of 2 points, and there are 5 points total. Therefore, the number of edges is equal to the number of ways to choose any 5 points from the set of 5 that we have:

$$\binom{5}{2} = 10$$

Also, as the polygon in question is a rectangle, each of the 5 of the sides are the same length. Similarly, all the sides of the diagonals are the same length, as there is only one possible angle between any set of non-adjecent points.

Therefore, if we choose any pair of elements in S, we have $\binom{10}{2} = 45$ possibilities. We know that the two chosen segments have the same length if and only if they are both sides, or both diagonals.

There are $\binom{5}{2} = 10$ ways of choosing two sides.

There are $\binom{5}{2} = 10$ ways of choosing two diagonals.

Therefore, the total number of ways to choose two segments that have the same length is:

$$\frac{10+10}{45} = \frac{4}{9}.$$

Thus, the correct answer is **B**.

13. Jo and Blair take turns counting from 1 to one more than the last number said by the other person.

Jo starts by saying"1", so Blair follows by saying "1, 2". Jo then says "1, 2, 3", and so on.

What is the 53^{rd} number said?

- A 2
- B 3
- C 5
- D 6
- E 8

Solution(s):

The sequence $1, 2 \cdots n$ is said first after the n^{th} triangular number T_n, which is:

$$T_n = \frac{n(n+1)}{2}.$$

(The triangular number T_n is defined as being the sum of the numbers 1 to n. As for why it has this formula, there's a bunch of ways you can prove it using induction or arithmetic sequences. We'll leave that to you, though!)

Moving on, writing out some triangular numbers, we notice that $T_9 = 45$ is the closest triangular number less than 53. This means that the sequence $1, 2 \cdots 9$ is first said after

$$T_9 = \frac{9 \cdot 10}{2} = 45$$

numbers.

Therefore, the 46^{th} number starts this new sequence at 1, and counting forwards, we can see that the $53rd$ number is 7 more than this, with a value of 8.

Thus, the correct answer is **E**.

14. Define $a \clubsuit b = a^2 b - ab^2$. Which of the following describes the set of points (x, y) for which $x \clubsuit y = y \clubsuit x$?

- A — A finite set of points
- B — One line
- C — Two parallel lines
- D — Two intersecting lines
- E — Three lines

Solution(s):

If $x \clubsuit y = y \clubsuit x$, we know that

$$x^2 y - xy^2 = y^2 x - x^2 y$$

$$2x^2 y - 2xy^2 = 0$$

$$(x)(y)(x - y) = 0.$$

Thus, we know that if $x = 0, x - y = 0$, or $y = 0$, we have a solution. These three solutions each take the form of a line, and so, the set of points where $x \clubsuit y = y \clubsuit x$ is a set of three points.

Thus, the correct answer is **E**.

15. A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

- A 1
- B $\dfrac{\sqrt{6}}{2}$
- C $\sqrt{3}$
- D 2
- E $\dfrac{3\sqrt{2}}{2}$

Solution(s):

Let the side length of the equilateral triangle be s. Then, let its area be A. This would make $a = 3s$.

As such, a hexagon with side s would have 6 equilateral triangles, with side length s, making its area $6A$.

Therefore, the side length of the hexagon with area A is equal to $\dfrac{s}{\sqrt{6}}$. As such,

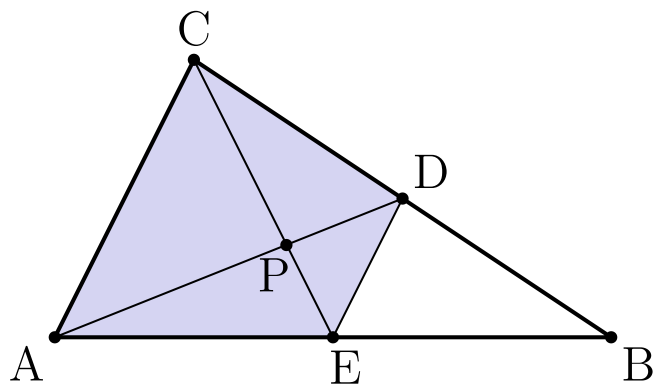
$$b = 6 \cdot \frac{s}{\sqrt{6}} = s\sqrt{6}.$$

This makes

$$\frac{a}{b} = \frac{3s}{s\sqrt{6}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}.$$

Thus, the correct answer is **B**.

16. In triangle $\triangle ABC$, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?



A 13

B 13.5

C 14

D 14.5

E 15

Solution(s):

Since we have an intersection of medians, we know that the segments are split at a $2:1$ ratio. Thus,

$$CP = 2PE = 3$$

and

$$AP = 2PD = 2(1.5) = 3$$

Then, since

$$\begin{aligned} PE^2 + PD^2 &= 1.5^2 + 2^2 \\ &= 2.5^2 \\ &= DE^2, \end{aligned}$$

the triangle $\triangle PED$ satisfies the Pythagorean Theorem, making $\angle DPE = 90^\circ$, and all of the angles at that point are therefore 90°.

Then, since the area of $AEDC$ is the area of all of the 4 right triangles put togeter, its area is

$$\begin{aligned} & \frac{CP \cdot AP + CP \cdot PD}{2} \\ + & \frac{PE \cdot AP + PE \cdot PD}{2} \\ = & \frac{(CP + PE)(AP + PD)}{2} \\ = & \frac{(4.5)(6)}{2} \\ = & \frac{27}{2} \\ = & 13.5. \end{aligned}$$

Thus, the correct answer is **B**.

17. Alex has 75 red tokens and 75 blue tokens. There is a booth where Alex can give two red tokens and receive in return a silver token and a blue token, and another booth where Alex can give three blue tokens and receive in return a silver token and a red token. Alex continues to exchange tokens until no more exchanges are possible. How many silver tokens will Alex have at the end?

- A 62
- B 82
- C 83
- D 102
- E 103

Solution(s):

Alex can do the following move: Take 6 red coins and going the the first booth, getting 3 blue coins and 3 silver coins, and getting 1 red coin and 4 silver coins total after this exchange from the second booth. He can do this as long as he has 6 red coins. Doing this 14 times yields 5 red coins, 75 blue coins, and 56 silver coins. Then, Alex can convert the red coins to blue, yielding 1 red coin, 77 blue coins, and 58 silver coins.

Then, Alex can do the following move: Take 6 blue coins and going the the second booth, getting 2 red coins and 2 silver coins, and getting 1 blue coin and 3 silver coins. Doing this 15 times yields 1 red coins, 2 blue coins, and 103 silver coins as we gain 45 coins.

Thus, the correct answer is **E**.

18. The number 2013 has the property that its units digit is the sum of its other digits, that is $2+0+1=3$. How many integers less than 2013 but greater than 1000 have this property?

A 33

B 34

C 45

D 46

E 58

Solution(s):

Given the first three numbers, if their sum is less than or equal to 9, it creates one number with the property.

Now, we can case on the 1st digit.

If it is 1, then the sum of the 2nd and 3rd digit must be less than or equal to 8. For each possible sum s, there are $s+1$ ways to choose the other numbers as the 2nd number can be anywhere from 0 to s.

Thus, the total is the 9^{th} triangular number:

$$\frac{9 \cdot 10}{2} = 45.$$

If it is 2, then the only way we can get a number that works less than 2013 is 2002, making a total of 46 cases.

Thus, the correct answer is **D**.

19. The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

A $-7 - 4\sqrt{3}$

B $-2 - \sqrt{3}$

C -1

D $-2 + \sqrt{3}$

E $-7 + 4\sqrt{3}$

Solution(s):

If there is exactly one root, let's call it r, we know that we can represent the quadratic as:

$$(x - r)^2 = x^2 - 2r + r^2$$

As such, we set $a = 1$, $b = -2r$, and $c = r^2$. Notice that this form of the quadratic is normalized, however, as we are finding roots (i.e. where the quadratic equals zero), we can scale the coefficients of the quadratic by any constant without changing the fundamental properties of the equation.

Now, as we know that c, b, a is a nonnegative arithmetic sequence, we know that there exists some constant k such that:

$$c + 2k = b + k = a$$

This suggests that $2b = c + a$, which we can plug in values to see that:

$$-4r = r^2 + 1$$

$$r^2 + 4r + 1 = 0$$

$$r = -2 \pm \sqrt{3}$$

Therefore, as we know that $b \geq c \geq 0$, we know that as $r \leq 0$ and $-2r \geq r^2 \iff r \geq -2$, so $r = -2 + \sqrt{3}$.

Thus, the correct answer is **D**.

20. The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!...a_m!}{b_1!b_2!...b_n!},$$

where

$$a_1 \geq a_2 \geq \cdots \geq a_m$$

and

$$b_1 \geq b_2 \geq \cdots \geq b_n$$

are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

A 1

B 2

C 3

D 4

E 5

Solution(s):

Let's start by looking at the prime factorization of 2013, which is $2013 = 61 \cdot 11 \cdot 3$. In order to have a factor of 61 in the numerator and to minimize a_1, we must set $a_1 = 61$.

Now, let's consider the denominator. For the numerator to have a factor of 61, we must have a factor of 61 in the denominator as well, because otherwise it would not be canceled out. Since we want to minimize b_1, we want the largest possible prime less than 61 in the denominator, which is 59. Therefore, $b_1 = 59$.

Finally, we can compute

$$|a_1 - b_1| = |61 - 59| = 2$$

Thus, the correct answer is **B**.

21. Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N ?

- A 55
- B 89
- C 104
- D 144
- E 273

Solution(s):

For the first sequence, let the first number be a_1 and let the second number be a_2.

Then, the sequence would be

$$a_1, a_2, a_1 + a_2, a_1 + 2a_2,$$

$$2a_1 + 3a_2, 3a_1 + 5a_2, 5a_1 + 8a_2$$

with $a_2 \geq a_1$.

If we had the second sequence start with b_1, b_2, we would get that:

$$5a_1 + 8a_2 = 5b_1 + 8b_2.$$

Thus,

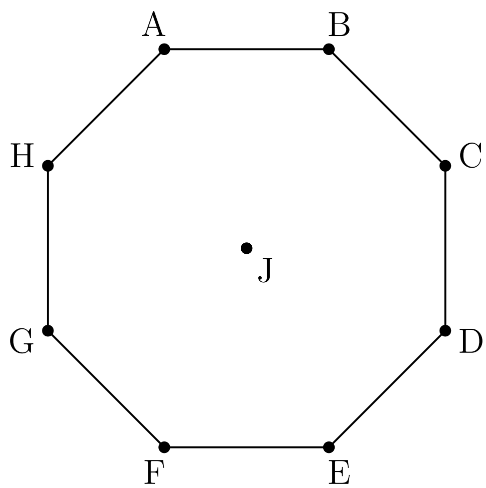
$$a_1 \equiv b_1 \mod 8,$$

$$a_2 \equiv b_2 \mod 5.$$

If we let $a_1 < b_1$, then we know $b_1 \geq 8$. As such, $b_2 \geq 8$. If we just let $b_1 = 8, b_2 = 8$, we can take $a_1 = 0, a_2 = 13$ and get $N = 104$ as a minimum possible answer.

Thus, the correct answer is **C**.

22. The regular octagon $ABCDEFGH$ has its center at J. Each of the vertices and the center are to be associated with one of the digits 1 through 9, with each digit used once, in such a way that the sums of the numbers on the lines AJE, BJF, CJG, and DJH are all equal. In how many ways can this be done?



A 384

B 576

C 1152

D 1680

E 3456

Solution(s):

Let S be defined as:

$$\begin{aligned} S &= A + J + E \\ &= B + J + F \\ &= C + J + G \\ &= D + J + H \end{aligned}$$

$$\begin{aligned} 4S = A + B + C + D + E \\ + F + G + H + 4J \end{aligned}$$

$$4S = 45 + 3J$$

$$45 + 3J \equiv 0 \mod 4$$

$$3J \equiv 3 \mod 4$$

$$J \equiv 1 \mod 4$$

This means that $J = 1, 5, 9$. From here, let's assume $J = 1$. We will see that the other cases are similar enough to omit.

If $J = 1$, then we know that the pairs of numbers that satisfy the equality above are:

$$2 + 9 = 3 + 8 = 4 + 7 = 5 + 6$$

There are $4!$ ways to distribute the pairs over the four groups, and then 2^4 ways for these groups to swap elements (i.e. $2 + 9 \iff 9 + 2$).

Now, if we look at the $J = 5$ and $J = 9$ cases, we see a similar pattern in the number of groupings and swaps. As such, we have:

$$3 \cdot 4! \cdot 2^4 = 1152$$

possibilities.

Thus, the correct answer is **C**.

23. In triangle $\triangle ABC$, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

A 18

B 21

C 24

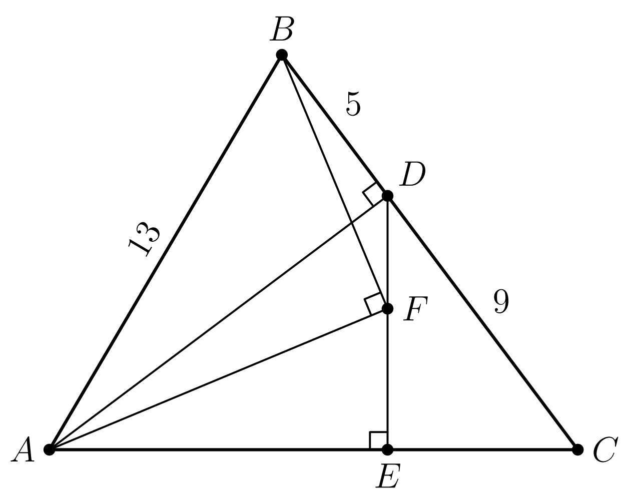
D 27

E 30

Solution(s):

First, we can deduce that $BD = 5, AD = 12, CD = 9$ by inspecting Pythagorean triples.

This yields the following diagram:



Then, we get $\angle ADE = \angle ACD$ by the similarity of $\triangle ADC$ and $\triangle AED$. Since $\triangle ABF$ and $\triangle ADF$ are both right triangles, they both have circumcircles with diameter AB, making $ABDF$ cyclic. Thus,

$$\angle ABF = \angle ADF = \angle ACD,$$

making

$$\cos(\angle ABF) = \frac{3}{5},$$

$$\sin(\angle ABF) = \frac{4}{5}.$$

As such,

$$AF = \frac{3}{5} \cdot 13,$$

$$BF = \frac{4}{5} \cdot 13.$$

By Ptolemy's Theorem, we get

$$AB \cdot DF + DB \cdot AF = BF \cdot AD.$$

Therefore,

$$13 \cdot DF + 5 \cdot 13\frac{4}{5} = 12 \cdot 13\frac{3}{5}.$$

This makes

$$DF + 5 \cdot \frac{4}{5} = 12 \cdot \frac{3}{5},$$

so $DF + 4 = \frac{36}{5}$. As such, $DF = \frac{16}{5}$, making $m + n = 21$.

Thus, the correct answer is **B**.

24. A positive integer n is "nice" if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set

$$\{2010, 2011, 2012, \ldots, 2019\}$$

are nice?

A 1

B 2

C 3

D 4

E 5

Solution(s):

A positive integer has 4 divisors if it can be written as p_1p_2 or p_1^3. The sum of the divisors would be

$$(p_1+1)(p_2+1)$$

or

$$p_1^3+p_1^2+p_1+1.$$

The second option can't happen since the value for $p_1=11$ is less than 2010 and the value for $p_1=13$ is greater.

Thus, we have $(p_1+1)(p_2+1)$ as our factored form.

Now, if either p_1 or p_2 are 2, then looking at $(p_1+1),(p_2+1)$ we know that one of them must be equal to 3, and other is still even. The elements of the set that satisfy this are even multiples of three, or 2010 and 2016.

However, observe that:

$$\frac{2010}{3}-1=669$$

$$\frac{2016}{3}-1=671$$

These aren't prime, so we conclude that p_1 and p_2 must be odd primes. This implies that $p_1 + 1$ and $p_2 + 1$ are both even, and as such, the number in question must be divisible by 4, giving only 2012 and 2016 as valid options.

Looking at each of these options:

$$2012 = 4 \cdot 503$$

Which implies $p_1 = 3, p_2 = 502$ which isn't a prime.

$$2016 = 4 \cdot 504$$

Which implies that $p_1 = 3, p_2 = 503$, which are both odd primes.

Therefore, the only "nice" number in the set is 2016.

Thus, the correct answer is **A**.

25. Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S.

For example, if $N = 749$, Bernardo writes the numbers $10{,}444$ and $3{,}245$, and LeRoy obtains the sum $S = 13{,}689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?

A 5

B 10

C 15

D 20

E 25

Solution(s):

First, we inspect on the units digit. If we let

$$N \equiv a \mod 5,$$

$$N \equiv b \mod 6,$$

we can set the units digit of S and N equal to get that

$$a + b \equiv 2a \mod 10$$

$$b \equiv a \mod 10.$$

Since $a < 10, b < 10$ we conclude that $a = b$.

This makes $N \equiv a \mod 30$ for $a < 5$.

Then, suppose we have $N = 30x + a$. If we have $a = 0$ working, then $a = 0, 1, 2, 3, 4$ works and vice versa, so we need only to check $a = 0$, and for all $a = 0$ that works, we can just multiply by 5.

Thus, we need only to look at multiples of 30.

Then, let

$$N = 36a + 6b,$$

$$N = 25x + 5y.$$

We then know

$$\begin{aligned} 10b + 10y &\equiv 2N \mod 100 \\ &\equiv 50x + 10y \\ &\equiv 72a + 12b \end{aligned}$$

Suggesting that:

$$10b \equiv 50x \mod 100.$$

As such, b is a multiple of 5 and $b + y$ must be even.

Therefore, given any y, we have a unique possible solution as there are two possible solutions for b given any y since b must be 0 or 5 and only one of them yields $y + b$ is even.

So, for each of the 5 possible y, we have a unique b that works, and for each y we have 5 values for a.

Thus, for each remainder of N when divided by 25, there is a unique remainder when divided by 36, making 25 out of the possible remainders when divided by 900 a solution.

From 100 to 999, each of the remainders appear exactly once, so there are 25 solutions.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc10/2013B