2014 AMC 10A Solutions

Typeset by: LIVE, by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2014A/solutions

1. What is

$$10 \cdot \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10}\right)^{-1}?$$

- A 3
- B 8
- C $\frac{25}{2}$
- D $\frac{170}{3}$
- E 170

Solution(s):

We get that

$$\begin{aligned} &\frac{1}{2} + \frac{1}{5} + \frac{1}{10} \\ &= \frac{5}{10} + \frac{2}{10} + \frac{1}{10} \\ &= \frac{4}{5}. \end{aligned}$$

Then $\left(\frac{4}{5}\right)^{-1} = \frac{5}{4}$ and then finally,

$$10 \cdot \frac{5}{4} = \frac{25}{2}.$$

Thus, **C** is the correct answer.

2. Roy's cat eats $\frac{1}{3}$ of a can of cat food every morning and $\frac{1}{4}$ of a can of cat food every evening. Before feeding his cat on Monday morning, Roy opened a box containing 6 cans of cat food. On what day of the week did the cat finish eating all the cat food in the box?

A Tuesday

B Wednesday

C Thursday

D Friday

E Saturday

Solution(s):

The cat eats

$$\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$$

cans of food each day. This means that it will take the cat

$$6 \div \frac{7}{12}=\frac{72}{7}$$

days to finish all the foods. Note that this value is between 10 and 11.

This means that the cat will finish eating the food in 11 days, which is 10 days after Monday.

10 days after Monday is the same as 3 days after Monday, and as such, the answer is Thursday.

Thus, **C** is the correct answer.

3. Bridget bakes 48 loaves of bread for her bakery. She sells half of them in the morning for \$ 2.50 each. In the afternoon she sells two thirds of what she has left, and because they are not fresh, she charges only half price. In the late afternoon she sells the remaining loaves at a dollar each. Each loaf costs \$ 0.75 for her to make. In dollars, what is her profit for the day?

A 24

B 36

C 44

D 48

E 52

Solution(s):

In the morning, she sells $48 \div 2 = 24$. From this, she makes 24 \cdot \$ 2.50 = \$60. In the afternoon, she has 24 loaves left, $24 \cdot \dfrac{2}{3} = 16$ of which she sells for 16 \cdot \dfrac{\$ 2.50}{2} = \$20. Finally, she sells the remaining $24 - 16 = 8$ loaves for a dollar each, for a total of \$ 8.

Her total revenue for the day is \$ 60 + \$ 20 + \$ 8 = \$ 88. The cost of all the loaves is 48 \cdot \$ 0.75 = \$ 36. Her total profits are then \$88 - \$36 = \$52.

Thus, **E** is the correct answer.

4. Walking down Jane Street, Ralph passed four houses in a row, each painted a different color. He passed the orange house before the red house, and he passed the blue house before the yellow house. The blue house was not next to the yellow house. How many orderings of the colored houses are possible?

A 2

B 3

C 4

D 5

E 6

Solution(s):

There are only two choices for the position of the yellow house, the third or fourth spot.

Case 1 : the yellow house is in the third spot

This forces the blue house to be the first, and this makes the orange house second and the red house fourth.

Case 2 : the yellow house is the last house

If the blue house is first, then the orange house is second, and the red house is third.

If the blue house is second, then the orange house is first, and the red house is second.

This gives us 3 possible orderings for the houses that satisfy all the conditions.

Thus, **B** is the correct answer.

5. On an algebra quiz, 10% of the students scored 70 points, 35% scored 80 points, 30% scored 90 points, and the rest scored 100 points. What is the difference between the mean and median score of the students' scores on this quiz?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution(s):

We can assign the number of students as 20 since this value will not affect the answer.

Then we have that 2 students got 70 points, 7 students got 80, 6 got 90, and 5 got 100.

We get that the mean is

$$\frac{2 \cdot 70 + 7 \cdot 70 + 6 \cdot 90 + 5 \cdot 100}{20}$$
$$= \frac{1740}{20}$$
$$= 87.$$

We also get that the median is 90, since 9 students got below a 90, and the 10 th and 11 th students got 90.

The difference between the two is $90 - 87 = 3$.

Thus, **C** is the correct answer.

6. Suppose that a cows give b gallons of milk in c days. At this rate, how many gallons of milk will d cows give in e days?

A $\dfrac{bde}{ac}$

B $\dfrac{ac}{bde}$

C $\dfrac{abde}{c}$

D $\dfrac{bcde}{a}$

E $\dfrac{abc}{de}$

Solution(s):

We have to multiply b by $\frac{d}{a}$ to account for the new number of cows.

We then have to multiply by $\frac{e}{c}$ to account for the new time that we have.

This gives us a final answer of

$$b \cdot \frac{d}{a} \cdot \frac{e}{c} = \frac{bde}{ac}.$$

Thus, **A** is the correct answer.

7. Nonzero real numbers x, y, a, and b satisfy $x < a$ and $y < b$. How many of the following inequalities must be true?

(I) $x + y < a + b$

(II) $x - y < a - b$

(III) $xy < ab$

(IV) $\frac{x}{y} < \frac{a}{b}$

A 0

B 1

C 2

D 3

E 4

Solution(s):

Adding the two inequalities together gets us

$$x + y < a + b,$$

which shows that **(I)** is correct.

One cannot subtract inequalities, which means that **(II)** is not necessarily true.

Consider $x = 1$, $y = 1$, $a = 2$, and $b = 3$ as a counter-example. This would give us $0 < -1$.

(III) is also not always true, since x and y might be negative numbers.

Let $x = -3$, $y = -2$, $a = 1$, and $b = 1$. Then $xy = 6$ and $ab = 1$ which shows that **(III)** is wrong.

The same thing occurs with **(IV)**. Using the same values as above, we have $\frac{x}{y} = 1.5$ and $\frac{a}{b} = 1$.

This shows that **(I)** is the only true statement.

Thus, **B** is the correct answer.

8. Which of the following numbers is a perfect square?

A $\dfrac{14!15!}{2}$

B $\dfrac{15!16!}{2}$

C $\dfrac{16!17!}{2}$

D $\dfrac{17!18!}{2}$

E $\dfrac{18!19!}{2}$

Solution(s):

Note that all of these answer choices are of the form

$$\frac{n!(n+1)!}{2} = \frac{(n!)^2(n+1)}{2}.$$

We have that $(n!)^2$ is square, so we need $\dfrac{n+1}{2}$ to be square as well.

This means that $n+1$ must be twice a perfect square. The only choice we have is $n+1=18$, which gives us $n=17$.

Thus, **D** is the correct answer.

9. The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle?

A 1

B 2

C 3

D 4

E 5

Solution(s):

We get that the area of the triangle is

$$\frac{1}{2} \cdot 2\sqrt{3} \cdot 6 = 6\sqrt{3}.$$

The length of the hypotenuse is

$$\sqrt{(2\sqrt{3})^2 + 6^2} = \sqrt{48} = 4\sqrt{3}.$$

Dropping the altitude, h, from the vertex to the hypotenuse, we get that

$$\frac{1}{2} \cdot h \cdot 4\sqrt{3} = 6\sqrt{3}$$

$$h = 3.$$

Thus, **C** is the correct answer.

10. Five positive consecutive integers starting with a have average b. What is the average of 5 consecutive integers that start with b?

A $a+3$

B $a+4$

C $a+5$

D $a+6$

E $a+7$

Solution(s):

Note that the average of 5 consecutive numbers starting with x is

$$\frac{5x+1+2+3+4}{5}=x+2.$$

This means that the average of 5 consecutive integers starting with a is $a+2$, which we know is b.

Furthermore, the average of 5 consecutive numbers starting with b is

$$b+2=a+4.$$

Thus, **B** is the correct answer.

11. A customer who intends to purchase an appliance has three coupons, only one of which may be used:

Coupon 1 : 10% off the listed price if the listed price is at least $50

Coupon 2 : $ 20 off the listed price if the listed price is at least $100

Coupon 3 : 18% off the amount by which the listed price exceeds $100

For which of the following listed prices will coupon 1 offer a greater price reduction than either coupon 2 or coupon 3?

A $ 179.95

B $ 199.95

C $ 219.95

D $ 239.95

E $ 259.95

Solution(s):

Let us analyze what these coupons do to an arbitrary price, x.

Coupon 1 changes this price to $.9x$. Coupon 2 changes the price to $x-20$. Coupon 3 changes the price to

$$x-.18(x-100)=.82x+18.$$

We want

$$.9x<x-20$$

and

$$.9x<.82x+18.$$

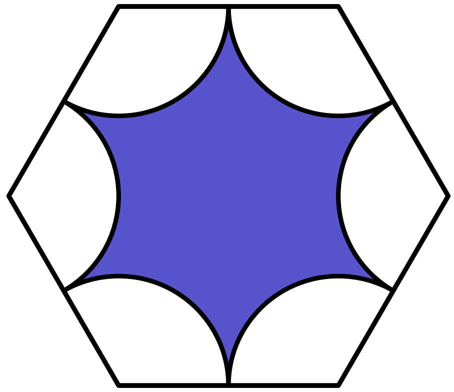
Solving both gives us

$$200<x<225.$$

The only answer choice that works is $ 219.95.

Thus, **C** is the correct answer.

12. A regular hexagon has side length 6. Congruent arcs with radius 3 are drawn with the center at each of the vertices, creating circular sectors as shown. The region inside the hexagon but outside the sectors is shaded as shown What is the area of the shaded region?



A $27\sqrt{3} - 9\pi$

B $27\sqrt{3} - 6\pi$

C $54\sqrt{3} - 18\pi$

D $54\sqrt{3} - 12\pi$

E $108\sqrt{3} - 9\pi$

Solution(s):

Note that we can split the hexagon up into 6 equilateral triangles each with side length 6.

Recall that the area of an equilateral triangle with side length s

$$\frac{s^2\sqrt{3}}{4}.$$

This means that the area of the hexagon is

$$6 \cdot \frac{6^2\sqrt{3}}{4} = 54\sqrt{3}.$$

Since each interior angle of a regular hexagon is 120°, the six sectors form 2 full circles.

This means that the area of all the sectors is

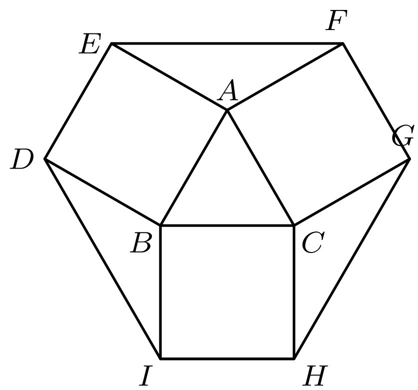
$$2 \cdot 3^2\pi = 18\pi.$$

The area of the shaded region is then

$$54\sqrt{3} - 18\pi.$$

Thus, **C** is the correct answer.

13. Equilateral $\triangle ABC$ has side length 1, and squares $ABDE$, $BCHI$, $CAFG$ lie outside the triangle. What is the area of hexagon $DEFGHI$?



A $\frac{12+3\sqrt{3}}{4}$

B $\frac{9}{2}$

C $3+\sqrt{3}$

D $\frac{6+3\sqrt{3}}{2}$

E 6

Solution(s):

We can find the areas of all the individual pieces and then add them up together.

The area of the center equilateral triangle is

$$\frac{1^2\sqrt{3}}{4} = \frac{\sqrt{3}}{4}.$$

We have that the areas of all the squares is

$$3 \cdot 1^2 = 3.$$

We also have that

$$\begin{aligned}\angle EAF &= 360^\circ - 60^\circ - 2 \cdot 90^\circ \\ &= 120^\circ.\end{aligned}$$

We also have that $\triangle EAF$ is isosceles, which means that we can rearrange the triangle by splitting it down the middle and recombining it into an equilateral triangle.

This means that the area of the three triangles is then

$$3 \cdot \frac{1^2\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}.$$

The total area is then

$$\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} + 3 = 3 + \sqrt{3}.$$

Thus, **C** is the correct answer.

14. The y-intercepts, P and Q, of two perpendicular lines intersecting at the point $A(6, 8)$ have a sum of zero. What is the area of $\triangle APQ$?

A 45

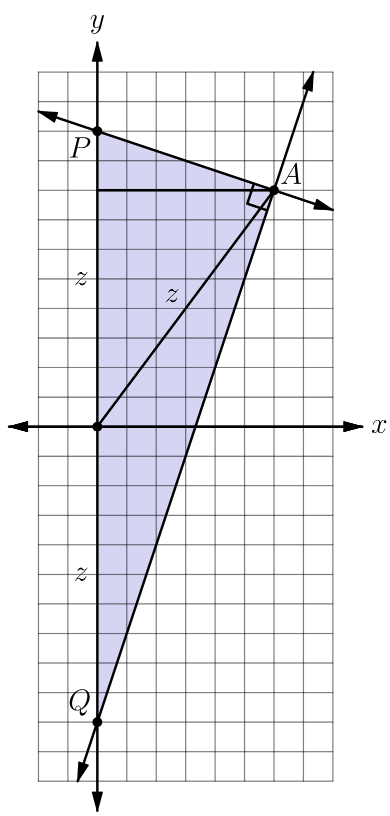
B 48

C 54

D 60

E 72

Solution(s):



We have that the y-intercepts are an equal distance from the origin since their values sum to 0.

Let this distance be z. We also have that that the distance from A to the origin is z since it is the median to the midpoint of the hypotenuse.

We then know that

$$z = \sqrt{6^2 + 8^2} = 10$$

by the distance formula. We know the altitude from A to $\overline{PQ}$ is 6 (it is just the x-value of A).

We also know that $PQ = 2 \cdot 10 = 20$, which tells us that the area

$$[APQ] = \frac{1}{2} \cdot 6 \cdot 20 = 60.$$

Thus, **D** is the correct answer.

15. David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

A 140

B 175

C 210

D 245

E 280

Solution(s):

Note that David drives at 50 miles per hour after one hour. Let the distance he still needs to be drive be d.

Then, if the airport is x miles from David's house, we know that:

$$\frac{x}{35} - \left(1 + \frac{x-35}{50}\right) = \frac{3}{2}$$

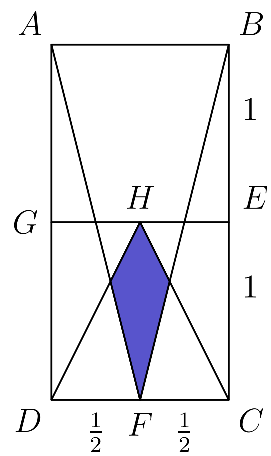
We solve this equation as follows:

$$\begin{aligned} \frac{x}{35} - \left(1 + \frac{x-35}{50}\right) &= \frac{3}{2} \\ \frac{x}{35} - \frac{x-35+50}{50} &= \frac{3}{2} \\ 10x - 7(x+15) &= 525 \\ 10x - 7x - 105 &= 525 \\ 3x &= 630 \\ x &= 210 \end{aligned}$$

Therefore, the airport is $x = 210$ miles from David's house.

Thus, **C** is the correct answer.

16. In rectangle $ABCD$, $AB = 1$, $BC = 2$, and points E, F, and G are midpoints of $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point H is the midpoint of $\overline{GE}$. What is the area of the shaded region?



A $\dfrac{1}{12}$

B $\dfrac{\sqrt{3}}{18}$

C $\dfrac{\sqrt{2}}{12}$

D $\dfrac{\sqrt{3}}{12}$

E $\dfrac{1}{6}$

Solution(s):

We can find the area of the shaded region by finding the area of $\triangle DHC$ and subtracting out the two unshaded triangles.

Extend $\overline{DH}$ so that it hits B. Let the intersection of $\overline{DB}$ and $\overline{AF}$ be X.

We have that $\triangle DXF \sim \triangle BXA$. Since $AB = 2 \cdot DF$, we have that $BX = 2 \cdot AX$.

This means that $DX = \dfrac{1}{3} \cdot DB$, which means that the altitude of $\triangle DXF$ is $\dfrac{1}{3}$ the height of the rectangle.

The area of $\triangle DXF$ is then

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6}.$$

The area of both unshaded triangles is then $2 \cdot \frac{1}{6} = \frac{1}{3}$. The area of $\triangle DHC$ is

$$\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.$$

The area of the shaded region is then $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

Thus, **E** is the correct answer.

17. Three fair six-sided dice are rolled. What is the probability that the values shown on two of the dice sum to the value shown on the remaining die?

A $\frac{1}{6}$

B $\frac{13}{72}$

C $\frac{7}{36}$

D $\frac{5}{24}$

E $\frac{2}{9}$

Solution(s):

Note that if one die is the sum of the other two dice, then it is strictly greater than the other two dice.

There are 3 ways to choose which of the dice is the sum of the other two, which makes it the greatest.

This die cannot be 1, since there is no way to sum two positive integers to get 1.

There is a $\frac{1}{6}$ chance that this die is any of the other numbers.

There is 1 way to get a sum of 2, 2 ways for 3, 3 for 4, 4 for 5, and 5 for 6.

We have take these numbers of ways out of a total of $6^2 = 36$ possibilities. The desired probability is then

$$3 \cdot \frac{1}{6} \cdot \frac{1+2+3+4+5}{36} = \frac{5}{24}.$$

Thus, **D** is the correct answer.

18. A square in the coordinate plane has vertices whose y-coordinates are 0, 1, 4, and 5. What is the area of the square?

A 16

B 17

C 25

D 26

E 27

Solution(s):

Let the points be

$$A = (a, 0,)$$

$$B = (b, 1),$$

$$C = (c, 4),$$

and

$$D = (d, 5).$$

Note that the difference in y-coordinates of A and C is 4.

As the angles of a square are right, we have that the difference in x-coordinates of A and B must be 4.

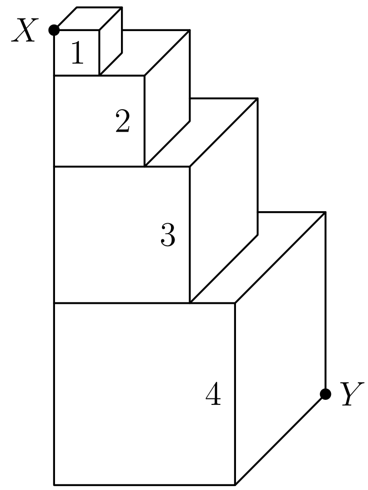
Using the distance formula, we get that

$$AB = \sqrt{4^2 + 1^2} = \sqrt{17}.$$

Squaring this tells us that the square's area is 17.

Thus, **B** is the correct answer.

19. Four cubes with edge lengths 1, 2, 3, and 4 are stacked as shown. What is the length of the portion of $\overline{XY}$ contained in the cube with edge length 3?



A $\dfrac{3\sqrt{33}}{5}$

B $2\sqrt{3}$

C $\dfrac{2\sqrt{33}}{3}$

D 4

E $3\sqrt{2}$

Solution(s):

The distance between X and Y with respect to the z-axis is

$$1+2+3+4=10.$$

Both the distances along the x and y-axes are 4.

Then

$$XY=\sqrt{4^2+4^2+10^2}=2\sqrt{33}.$$

Let the desired length be x. Then using similar triangles, we have that

$$\frac{x}{3}=\frac{2\sqrt{33}}{10}$$

$$x = \frac{3\sqrt{33}}{5}.$$

Thus, **A** is the correct answer.

20. The product $(8)(888\dots8)$, where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k?

A 901

B 911

C 919

D 991

E 999

Solution(s):

To see if any pattern exists, we can test out small values of k.

We have that

$$8 \cdot 8 = 64,$$

$$8 \cdot 88 = 704,$$

$$8 \cdot 888 = 7104,$$

$$8 \cdot 8888 = 71104.$$

From this, it is pretty safe to guess that for every increment of k, there is an extra 1 added to the product. If you know how to use induction, you can prove that this pattern holds, but that's not necessary to solve the problem.

This means that for any $k \geq 3$, the sum of the digits in the product is

$$7 + 4 + 0 + k - 2 = k + 9.$$

Finally, we get

$$k + 9 = 1000$$

$$k = 991.$$

Thus, **D** is the correct answer.

21. Positive integers a and b are such that the graphs of $y = ax + 5$ and $y = 3x + b$ intersect the x-axis at the same point. What is the sum of all possible x-coordinates of these points of intersection?

A -20

B -18

C -15

D -12

E -8

Solution(s):

Note that the lines intersect the x-axis when $y = 0$. This gives us

$$0 = ax + 5$$

and

$$0 = 3x + b,$$

which when solved gives us

$$x = -\frac{5}{a}$$

and

$$x = -\frac{b}{3}.$$

Setting these equal to each other, we have

$$\frac{5}{a} = \frac{b}{3}$$

$$ab = 15.$$

We know that a and b are positive, which means that the only pairs of values (a, b) that satisfy the above equation are

$$(1, 15),$$

$$(3, 5),$$

$$(5, 3),$$

$$(15, 1).$$

Plugging these values back into the equations gives us x-values of

$$x = -5, -\frac{5}{3}, -1, -\frac{1}{3}.$$

The sum of all these values is -8.

Thus, **E** is the correct answer.

22. In rectangle $ABCD$, $\overline{AB} = 20$ and $\overline{BC} = 10$. Let E be a point on $\overline{CD}$ such that $\angle CBE = 15^\circ$. What is $\overline{AE}$?

A $\frac{20\sqrt{3}}{3}$

B $10\sqrt{3}$

C 18

D $11\sqrt{3}$

E 20

Solution(s):

To make working with 15° easier, we can create a 30° angle and apply the angle bisector theorem.

Let F be the point on $\overline{DC}$ such that $\angle EBF = 15^\circ$. Applying the angle bisector theorem, we get

$$\frac{BC}{BF} = \frac{CE}{EF}.$$

Using the special right triangle properties of a $30-60-90$ triangle, we have that

$$CF = \frac{10\sqrt{3}}{3}$$

and

$$BF = \frac{20\sqrt{3}}{3}.$$

We can substitute in some values to get

$$\frac{10}{\frac{20\sqrt{3}}{3}} = \frac{CE}{EF}$$

$$\frac{2\sqrt{3}}{3}CE = EF.$$

Using $CE + EF = CF$, we have

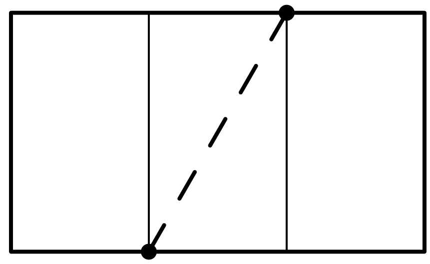
$$\frac{2\sqrt{3}}{3}CE + CE = \frac{10\sqrt{3}}{3}$$

$$CE = 20 - 10\sqrt{3}.$$

This means that $DE = 10\sqrt{3}$, which gives us $AE = 20$ since $\triangle ADE$ is also a $30 - 60 - 90$ triangle.

Thus, **E** is the correct answer.

23. A rectangular piece of paper whose length is $\sqrt{3}$ times the width has area A. The paper is divided into three equal sections along the opposite lengths, and then a dotted line is drawn from the first divider to the second divider on the opposite side as shown. The paper is then folded flat along this dotted line to create a new shape with area B. What is the ratio $\frac{B}{A}$?



A $\frac{1}{2}$

B $\frac{3}{5}$

C $\frac{2}{3}$

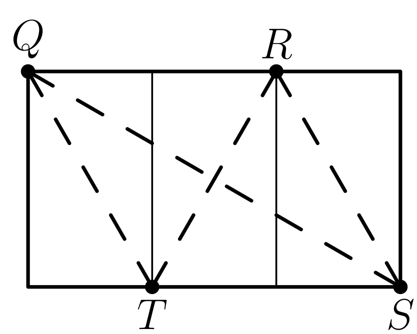
D $\frac{3}{4}$

E $\frac{4}{5}$

Solution(s):

WLOG, let the width of the rectangle be 1 and the length be $\sqrt{3}$.

Draw the line perpendicular to the midpoint of the fold, as shown below.



Note that $QR = \dfrac{2\sqrt{3}}{3}$ and

$$QT = \sqrt{1^2 + \left(\frac{\sqrt{3}}{3}\right)^2}$$
$$= \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{4}{3}}.$$

This tells us

$$QT = \frac{2\sqrt{3}}{3} = QR.$$

This means that $\triangle QRT$ is equilateral. Similarly, $\triangle RTS$ is equilateral. This makes the two triangles congruent.

This means that after the rectangle gets folded, this area will be overlapped. The area of the rectangle is $1 \cdot \sqrt{3} = \sqrt{3}$. The side length of this triangle is $QT = \dfrac{2\sqrt{3}}{3}$. The area of it is then

$$\left(\frac{2\sqrt{3}}{3}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{3}.$$

The area of the folded figure is then

$$\sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}.$$

The desired ratio is then

$$\frac{2\sqrt{3}}{3} \div \sqrt{3} = \frac{2}{3}.$$

Thus, **C** is the correct answer.

24. A sequence of natural numbers is constructed by listing the first 4, then skipping one, listing the next 5, skipping 2, listing 6, skipping 3, and on the nth iteration, listing $n+3$ and skipping n. The sequence begins

$$1, 2, 3, 4, 6, 7, 8, 9, 10, 13.$$

What is the $500,000$th number in the sequence?

A $996{,}506$

B $996{,}507$

C $996{,}508$

D $996{,}509$

E $996{,}510$

Solution(s):

We can just count the number of skipped numbers and add that on to $500{,}000$.

Note that

$$\frac{999 \cdot 1000}{2} < 500,000$$

and

$$500,000 < \frac{1000 \cdot 1001}{2}.$$

This means that there are $999 - 3 = 996$ skipped blocks of numbers in the sequence. We have to subtract 3 since we start off by listing 4 numbers and not 1.

Then,

$$n = \frac{996 \cdot 997}{2} = 496{,}506,$$

so the desired answer is

$$496{,}506 + 500{,}000 = 996{,}506.$$

Thus, **A** is the correct answer.

25. The number 5^{867} is between 2^{2013} and 2^{2014}. How many pairs of integers (m, n) are there such that $1 \leq m \leq 2012$ and

$$5^n < 2^m < 2^{m+2} < 5^{n+1}?$$

A 278

B 279

C 280

D 281

E 282

Solution(s):

Note that between any 2 consecutive powers of 5, there are either 1 or 2 powers of 2. This is because

$$2^2 < 5 < 2^3.$$

Consider the intervals from $(5^0, 5^1)$ to $(5^{866}, 5^{867})$.

Since

$$2^{2013} < 5^{867} < 2^{2014},$$

we know that these intervals must all together contain 2013 powers of 2.

Let x be the number of intervals that contain 2 powers of 2 and y contain 3 powers of 2.

Then

$$x + y = 867,$$
$$2x + 3y = 2013.$$

Multiplying the top equation by 2 and subtracting it from the bottom equation gives us $y = 279$.

Thus, **B** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2014A