

2018 AMC 10B Solutions

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1. Kate bakes a 20-inch by 18-inch pan of cornbread. The cornbread is cut into pieces that measure 2 inches by 2 inches. How many pieces of cornbread does the pan contain?

A 90

B 100

C 180

D 200

E 360

Solution:

The whole pan has area $20 \times 18 = 360$ square inches. Each piece is $2 \times 2 = 4$ square inches. So the number of pieces is $\frac{360}{4} = 90$. Thus, **A** is the correct answer.

2. Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

A 64

B 65

C 66

D 67

E 68

Solution:

Each leg is half an hour. In the first, Sam drove $60 \cdot \frac{1}{2} = 30$ miles; in the second, $65 \cdot \frac{1}{2} = 32.5$ miles. That's 62.5 miles so far. That leaves $96 - 62.5 = 33.5$ miles for the last half hour, which is a speed of $\frac{33.5}{\frac{1}{2}} = 67$ mph. Therefore, the answer is **D**.

3. In the expression $(_ \times _) + (_ \times _)$ each blank is to be filled in with one of the digits 1, 2, 3, or 4, with each digit being used once. How many different values can be obtained?

A 2

B 3

C 4

D 6

E 24

Solution:

Order inside a product doesn't matter, and neither does the order we add the two products. So all that matters is how the four digits split into two pairs. There are three splits: $1 \cdot 2 + 3 \cdot 4 = 14$, $1 \cdot 3 + 2 \cdot 4 = 11$, and $1 \cdot 4 + 2 \cdot 3 = 10$. That's 3 different values. Thus, **B** is the correct answer.

4. A three-dimensional rectangular box with dimensions X , Y , and Z has faces whose surface areas are 24, 24, 48, 48, 72, and 72 square units. What is $X + Y + Z$?

A 18

B 22

C 24

D 30

E 36

Solution:

The three distinct face areas are the pairwise products $XY = 24$, $XZ = 48$, $YZ = 72$ in some order. Multiply all three: $(XYZ)^2 = 24 \cdot 48 \cdot 72 = 82944$, so $XYZ = 288$. Now divide by each face area. We get $Z = \frac{288}{24} = 12$, $Y = \frac{288}{48} = 6$, and $X = \frac{288}{72} = 4$, so $X + Y + Z = 22$. Therefore, the answer is **B**.

5. How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

A 128

B 192

C 224

D 240

E 256

Solution:

Count the complement. The set has $2^8 = 256$ subsets total. A subset avoids every prime exactly when it sticks to the non-primes $\{4, 6, 8, 9\}$, and there are $2^4 = 16$ of those. So $256 - 16 = 240$ subsets contain at least one prime. Thus, **D** is the correct answer.

6. A box contains 5 chips, numbered 1, 2, 3, 4, and 5. Chips are drawn randomly one at a time without replacement until the sum of the values drawn exceeds 4. What is the probability that 3 draws are required?

A $\frac{1}{15}$

B $\frac{1}{10}$

C $\frac{1}{6}$

D $\frac{1}{5}$

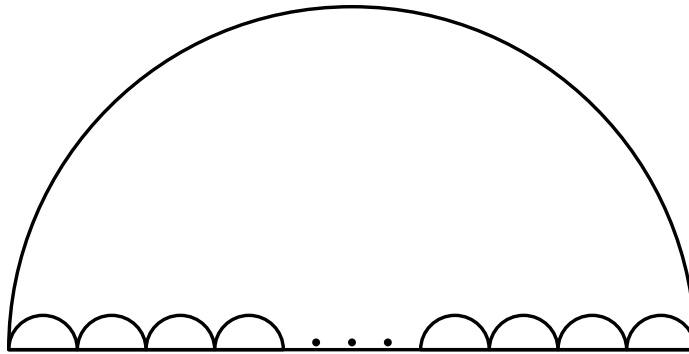
E $\frac{1}{4}$

Solution:

We need a third draw exactly when the first two chips still sum to 4 or less. The only such unordered pairs are $\{1, 2\}$ and $\{1, 3\}$. Each can be drawn in either order, giving 4 favorable ordered prefixes.

Imagine that a complete random ordering of all five chips is chosen in advance. Then all $5 \cdot 4 = 20$ ordered first-two-chip prefixes are equally likely, even when the actual process would stop after the first chip. Thus the probability is $\frac{4}{20} = \frac{1}{5}$. Therefore, the answer is **D**.

7. In the figure below, N congruent semicircles are drawn along a diameter of a large semicircle, with their diameters covering the diameter of the large semicircle with no overlap. Let A be the combined area of the small semicircles and B be the area of the region inside the large semicircle but outside the small semicircles. The ratio $A : B$ is $1 : 18$. What is N ?

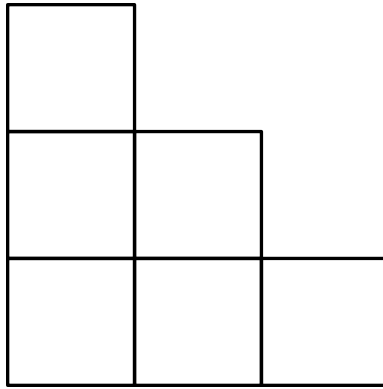


- ☐ A 16
- ☐ B 17
- ☐ C 18
- ☒ D 19
- ☐ E 36

Solution:

Let each small semicircle have radius r . The N diameters cover the big diameter, so the large radius is Nr . Then $A = N \cdot \frac{1}{2}\pi r^2$, and the large semicircle has area $\frac{1}{2}\pi(Nr)^2$, so the leftover region is $B = \frac{1}{2}\pi r^2(N^2 - N)$. This gives $A : B = N : N(N - 1) = 1 : (N - 1)$. Set $N - 1 = 18$, and $N = 19$. Thus, **D** is the correct answer.

8. Sara makes a staircase out of toothpicks as shown:



This is a 3-step staircase and uses 18 toothpicks. How many steps would be in a staircase that used 180 toothpicks?

- A 10
- B 11
- C 12**
- D 24
- E 30

Solution:

In an n -step staircase the vertical toothpicks number $(1 + 2 + \cdots + n) + n = \frac{n(n+1)}{2} + n$, and there are just as many horizontal ones. That's a total of $n(n + 1) + 2n = n(n + 3)$. Check: $n = 3$ gives 18, as it should. Now solve $n(n + 3) = 180$. This factors as $(n - 12)(n + 15) = 0$, so $n = 12$. Therefore, the answer is **C**.

9. The faces of each of 7 standard dice are labeled with the integers from 1 to 6. Let p be the probability that when all 7 dice are rolled, the sum of the numbers on the top faces is 10. What other sum occurs with the same probability p ?

A 13

B 26

C 32

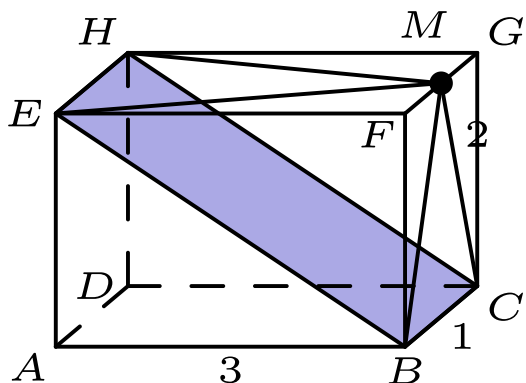
D 39

E 42

Solution:

Replace each die's value k by $7 - k$. This pairs up outcomes one-to-one and keeps their probabilities, and it sends a total of s to $7 \cdot 7 - s = 49 - s$. So the sums s and $49 - s$ are equally likely. The partner of 10 is $49 - 10 = 39$. Thus, **D** is the correct answer.

10. In the rectangular parallelepiped shown, $AB = 3$, $BC = 1$, and $CG = 2$. Point M is the midpoint of FG . What is the volume of the rectangular pyramid with base $BCH E$ and apex M ?



- ☐ A 1
- ☐ B $\frac{4}{3}$
- ☐ C $\frac{3}{2}$
- ☐ D $\frac{5}{3}$
- ☒ E 2

Solution:

Put A at the origin with edges along the axes: $A = (0, 0, 0)$, $B = (3, 0, 0)$, $C = (3, 1, 0)$, $E = (0, 0, 2)$, $H = (0, 1, 2)$, $F = (3, 0, 2)$, $G = (3, 1, 2)$, so $M = (3, \frac{1}{2}, 2)$. The base $BCH E$ is a rectangle with $BC = 1$ and $BE = \sqrt{3^2 + 2^2} = \sqrt{13}$, hence area $\sqrt{13}$. Its plane is $2x + 3z = 6$, and M sits at distance $\frac{|2 \cdot 3 + 3 \cdot 2 - 6|}{\sqrt{13}} = \frac{6}{\sqrt{13}}$ from it. The volume is $\frac{1}{3} \cdot \sqrt{13} \cdot \frac{6}{\sqrt{13}} = 2$. Therefore, the answer is **E**.

11. Which of the following expressions is never a prime number when p is a prime number?

A $p^2 + 16$

B $p^2 + 24$

C $p^2 + 26$

D $p^2 + 46$

E $p^2 + 96$

Solution:

Look at $p^2 + 26$. When $p = 3$, it's $35 = 5 \cdot 7$. For any other prime, p isn't divisible by 3, so $p^2 \equiv 1 \pmod{3}$ and $p^2 + 26 \equiv 1 + 2 \equiv 0 \pmod{3}$. Either way it's a multiple of 3 bigger than 3, hence composite. So it's never prime. Thus, **C** is the correct answer.

12. Line segment AB is a diameter of a circle with $AB = 24$. Point C , not equal to A or B , lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

A 25

B 38

C 50

D 63

E 75

Solution:

Put the center O at the origin, so $A = (-12, 0)$ and $B = (12, 0)$, while C runs over the circle of radius 12. Then $A + B = 0$, so the centroid is $\frac{1}{3}(A + B + C) = \frac{1}{3}C$. As C circles, $\frac{1}{3}C$ traces a circle of radius $\frac{12}{3} = 4$ (minus the two points where $C = A$ or B). Its area is $\pi \cdot 4^2 = 16\pi \approx 50$. Therefore, the answer is **C**.

13. How many of the first 2018 numbers in the sequence 101, 1001, 10001, 100001, ... are divisible by 101?

A 253

B 504

C 505

D 506

E 1009

Solution:

The k -th term is $10^{k+1} + 1$, which 101 divides iff $10^{k+1} \equiv -1 \pmod{101}$. Notice $10^2 = 100 \equiv -1 \pmod{101}$. So $10^m \equiv -1$ exactly when $m \equiv 2 \pmod{4}$, meaning $k + 1 \equiv 2$, that is $k \equiv 1 \pmod{4}$. Among $k = 1, 2, \dots, 2018$, the values 1, 5, ..., 2017 number 505. Thus, **C** is the correct answer.

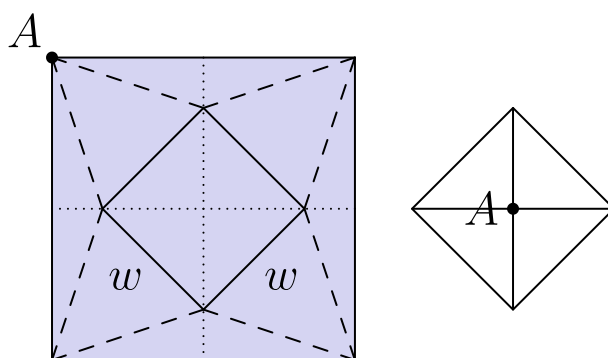
14. A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

- A 202
- B 223
- C 224
- D 225**
- E 234

Solution:

The mode shows up 10 times. To keep the number of distinct values small, let every other value repeat as much as the rules allow, which is 9 times each (any more would tie the mode). With d distinct values the list holds at most $10 + 9(d - 1)$ entries. We need $10 + 9(d - 1) \geq 2018$, so $d - 1 \geq 223.1$, giving $d \geq 225$. Therefore, the answer is **D**.

15. A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h . What is the area of the sheet of wrapping paper?



A $2(w + h)^2$

B $\frac{(w + h)^2}{2}$

C $2w^2 + 4wh$

D $2w^2$

E w^2h

Solution:

Let the sheet have side s . The base sits as a square of side w turned 45° , so the center is $\frac{w}{2}$ from each base edge. A corner of the sheet lies $\frac{s}{\sqrt{2}}$ from the center. Folding that corner up to the top center traces a straight line: $\frac{w}{2}$ out to the base edge, then h up the side, then $\frac{w}{2}$ across the top. So $\frac{s}{\sqrt{2}} = \frac{w}{2} + h + \frac{w}{2} = w + h$. Then $s = \sqrt{2}(w + h)$, and the area is $s^2 = 2(w + h)^2$. Thus, **A** is the correct answer.

16. Let $a_1, a_2, \dots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

- ☐ A 0
- ☐ B 1
- ☐ C 2
- ☐ D 3
- ☒ E 4

Solution:

For any integer n , $n^3 - n = (n - 1)n(n + 1)$ is a product of three consecutive integers, so it's divisible by 6. That means $n^3 \equiv n \pmod{6}$. Summing, $\sum a_i^3 \equiv \sum a_i = 2018^{2018} \pmod{6}$. Now $2018 \equiv 2 \pmod{6}$, and powers of 2 mod 6 alternate 2, 4, 2, 4, The exponent 2018 is even, so $2^{2018} \equiv 4 \pmod{6}$. The remainder is 4. Therefore, the answer is **E**.

17. In rectangle $PQRS$, $PQ = 8$ and $QR = 6$. Points A and B lie on PQ , points C and D lie on QR , points E and F lie on RS , and points G and H lie on SP so that $AP = BQ < 4$ and the convex octagon $ABCDEFGH$ is equilateral. The length of a side of this octagon can be expressed in the form $k + m\sqrt{n}$, where k , m , and n are integers and n is not divisible by the square of any prime. What is $k + m + n$?

- A 1
- B 7**
- C 21
- D 92
- E 106

Solution:

Let s be the octagon's side length and let $x = AP = BQ = \frac{8-s}{2}$. The right triangles $\triangle APH$ and $\triangle BQC$ have the same hypotenuse s and a leg of length x , so they are congruent; write $PH = QC = y$. Because $CD = HG = s$ and the vertical sides of the rectangle both have length 6, it follows that $DR = GS$. The right triangles at R and S are then congruent, so $RE = SF$. Since $RS = 8$ and $EF = s$, each of these equal lengths is $\frac{8-s}{2} = x$. Thus all four cut corners have legs x and y .

The equal octagon sides give

$$8 - 2x = 6 - 2y = \sqrt{x^2 + y^2}.$$

The first equality gives $y = x - 1$. Substituting and squaring gives $2x^2 - 30x + 63 = 0$, so the root with $x < 4$ is $x = \frac{15-3\sqrt{11}}{2}$.

The side length is $8 - 2x = -7 + 3\sqrt{11}$, so $k + m + n = -7 + 3 + 11 = 7$. Thus, **B** is the correct answer.

18. Three young brother-sister pairs from different families need to take a trip in a van. These six children will occupy the second and third rows in the van, each of which has three seats. To avoid disruptions, siblings may not sit right next to each other in the same row, and no child may sit directly in front of his or her sibling. How many seating arrangements are possible for this trip?

- A 60
- B 72
- C 92
- D 96**
- E 120

Solution:

Suppose some family put both children in one row. They'd have to take the non-adjacent seats 1 and 3, which forces the middle family's two children into the same column. Not allowed. So each row holds exactly one child from each family. The second row is a permutation of the three families, $3! = 6$ ways. The third row needs a different family in every column, a derangement of the second row's order, and there are 2 of those. Finally, each pair can swap its two children between their seats, $2^3 = 8$ ways. The total is $6 \cdot 2 \cdot 8 = 96$. Therefore, the answer is **D**.

19. Joey and Chloe and their daughter Zoe all have the same birthday. Joey is 1 year older than Chloe, and Zoe is exactly 1 year old today. Today is the first of the 9 birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

- A 7
- B 8
- C 9
- D 10
- E 11

Solution:

Let Chloe be n today; Zoe is 1. In t years their age ratio is

$$\frac{n+t}{1+t} = 1 + \frac{n-1}{1+t},$$

which is an integer exactly when $1+t$ divides $n-1$. Thus $n-1$ has exactly 9 positive divisors.

A number with 9 divisors has the form p^8 or p^2q^2 . The only two-digit possibility is $2^2 \cdot 3^2 = 36$, so Chloe is 37 and Joey is 38.

Joey's age $38+t$ is a multiple of Zoe's age $1+t$ exactly when $1+t$ divides 37. The next time is $t = 36$, when Joey is 74. Its digit sum is $7+4 = 11$. Thus, **E** is the correct answer.

20. A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

A 2016

B 2017

C 2018

D 2019

E 2020

Solution:

Notice $f(n) = n + 1$ solves the recurrence on its own, so write $f(n) = (n + 1) + g(n)$. Then g satisfies the homogeneous version $g(n) = g(n-1) - g(n-2)$. With $g(1) = -1$ and $g(2) = -2$, it cycles with period 6: $-1, -2, -1, 1, 2, 1, \dots$. Since $2018 \equiv 2 \pmod{6}$, we get $g(2018) = -2$, so $f(2018) = 2019 - 2 = 2017$.

Therefore, the answer is **B**.

21. Mary chose an even 4-digit number n . She wrote down all the divisors of n in increasing order from left to right: $1, 2, \dots, \frac{n}{2}, n$. At some moment Mary wrote **323** as a divisor of n . What is the smallest possible value of the next divisor written to the right of **323**?

A 324

B 330

C 340

D 361

E 646

Solution:

Let d be the next divisor after **323**. If $\gcd(d, 323) = 1$, then n is a multiple of $323d > 323^2 > 9999$, impossible for a four-digit number. Thus $\gcd(d, 323) > 1$.

Since $323 = 17 \cdot 19$, this gcd is at least 17. It also divides $d - 323$, so $d - 323 \geq 17$ and hence $d \geq 340$.

This bound is attained: for $n = 6460 = 2^2 \cdot 5 \cdot 17 \cdot 19$, both **323** and **340** are divisors. The lower bound shows there is no divisor between them. Thus the smallest possible next divisor is **340**, and **C** is the correct answer.

22. Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. Which of the following numbers is closest to the probability that x , y , and 1 are the side lengths of an obtuse triangle?

A 0.21

B 0.25

C 0.29

D 0.50

E 0.79

Solution:

The three lengths x , y , 1 make a triangle iff $x + y > 1$. Since 1 is the longest side, that triangle is obtuse iff $x^2 + y^2 < 1$. So in the unit square we want the region inside the quarter circle $x^2 + y^2 = 1$ but above the line $x + y = 1$. That's the quarter disk with the right triangle under the chord removed: $\frac{\pi}{4} - \frac{1}{2} \approx 0.285$. The closest choice is 0.29. Therefore, the answer is **C**.

23. How many ordered pairs (a, b) of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot \text{lcm}(a, b) + 12 \cdot \text{gcd}(a, b),$$

where $\text{gcd}(a, b)$ denotes the greatest common divisor of a and b , and $\text{lcm}(a, b)$ denotes their least common multiple?

- ☐ A 0
- ☒ B 2
- ☐ C 4
- ☐ D 6
- ☐ E 8

Solution:

Recall $ab = \text{gcd}(a, b) \text{lcm}(a, b)$. Let $x = \text{lcm}(a, b)$ and $y = \text{gcd}(a, b)$. The equation becomes

$$(x - 12)(y - 20) = 177 = 3 \cdot 59.$$

The positive factor pairs give $(x, y) = (13, 197), (189, 21), (15, 79)$, and $(71, 23)$. The negative factor pairs make either x or y negative, so they are impossible. Also y must divide x , and only $(x, y) = (189, 21)$ passes.

Write $a = 21u$ and $b = 21v$. Then $\text{gcd}(u, v) = 1$ and $uv = \frac{189}{21} = 9$, so $(u, v) = (1, 9)$ or $(9, 1)$. Hence the two ordered pairs are $(21, 189)$ and $(189, 21)$, and **B** is the correct answer.

24. Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y , and Z the midpoints of sides AB, CD , and EF , respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

A $\frac{3}{8}\sqrt{3}$

B $\frac{7}{16}\sqrt{3}$

C $\frac{15}{32}\sqrt{3}$

D $\frac{1}{2}\sqrt{3}$

E $\frac{9}{16}\sqrt{3}$

Solution:

The triangle XYZ is equilateral with side $\frac{3}{2}$, so its area is

$$\frac{\sqrt{3}}{4} \left(\frac{3}{2}\right)^2 = \frac{9\sqrt{3}}{16}.$$

The triangles ACE and XYZ are concentric and rotated 30° from each other. At each vertex of XYZ , the sides of ACE cut off a 30-60-90 triangle whose hypotenuse is the half-side segment $AX = \frac{1}{2}$. Its legs are $\frac{1}{4}$ and $\frac{\sqrt{3}}{4}$, so each corner has area $\frac{\sqrt{3}}{32}$.

Removing the three corners gives

$$\frac{9\sqrt{3}}{16} - 3 \cdot \frac{\sqrt{3}}{32} = \frac{15\sqrt{3}}{32}.$$

Therefore, the answer is **C**.

25. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . How many real numbers x satisfy the equation $x^2 + 10,000\lfloor x \rfloor = 10,000x$?

A 197

B 198

C 199

D 200

E 201

Solution:

Let $a = \lfloor x \rfloor$. The equation reads $x^2 = 10,000(x - a) = 10,000\{x\}$, and since $0 \leq \{x\} < 1$, this forces $0 \leq x^2 < 10,000$, so $-100 < x < 100$. On each interval $[a, a + 1)$ the quantity $10,000x - x^2$ increases from $10,000a - a^2$ and approaches, but does not reach, $10,000(a + 1) - (a + 1)^2$. It hits $10,000a$ exactly once precisely when $(a + 1)^2 < 10,000$. That holds for the integers $-100 \leq a \leq 98$, which is 199 solutions. Thus, **C** is the correct answer.

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