2020 AMC 10B Solutions

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https://live.poshenloh.com/past-contests/amc10/2020B/solutions

1. What is the value of

$$1-(-2)-3-(-4)-5-(-6)?$$

A -20

B -3

C 3

D 5

E 21

Solution(s):

Let's first notice that when we subtract a negative number, it is the same as adding it's positive counterpart. In other words:

$$a-(-b)=a+b$$

With that in mind, we can rewrite the above expression as:

$$1+2-3+4-5+6$$

Which can be solved to yield 5 as our final answer.

Thus, **D** is the correct answer.

2. Carl has 5 cubes each having side length 1, and Kate has 5 cubes each having side length 2. What is the total volume of the 10 cubes?

- A 24
- B 25
- C 28
- D 40
- E 45

Solution(s):

Recall that a cube with a side length of a has a volume of a^3. With this in mind, Carl's cubes (each with side length 1) have a volume of $1^3 = 1$ each. Therefore, all of Carl's 5 cubes will have a total volume of $5 \cdot 1 = 5$.

Similarly, Kate's cubes (each with side length 2) have a volume of $2^3 = 8$ each. Therefore, all of Kates's 5 cubes will have a total volume of $5 \cdot 8 = 40$.

Therefore, the total volume of these 10 cubes is $5 + 40 = 45$.

Thus, the correct answer is **E**.

3. The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of w to y?

- A $4:3$
- B $3:2$
- C $8:3$
- D $4:1$
- E $16:3$

Solution(s):

Let's begin by restating the following:

$$w:x = \frac{w}{x} = \frac{4}{3} = 4:3$$

$$y:z = \frac{y}{z} = \frac{3}{2} = 3:2$$

$$z:x = \frac{z}{x} = \frac{1}{6} = 1:6$$

Armed with these three equations, we want to find $w:y = \frac{w}{y}$. To do this, we must try and represent $\frac{w}{y}$ using $\frac{w}{x}$, $\frac{y}{z}$, and $\frac{z}{y}$.

Notice that we can represent $\frac{w}{y}$ as $\frac{w}{z} \cdot \frac{z}{y}$.

Further notice that $\frac{w}{z} = \frac{w}{x} \cdot \frac{x}{z}$. Therefore, combining these two facts shows us that:

$$\begin{aligned}
\frac{w}{y} &= \frac{w}{x} \cdot \frac{x}{z} \cdot \frac{z}{y} \\
&= \frac{w}{x} \cdot \frac{1}{(\frac{z}{x})} \cdot \frac{1}{(\frac{y}{z})} \\
&= \frac{4}{3} \cdot \frac{1}{(\frac{1}{6})} \cdot \frac{1}{(\frac{3}{2})} \\
&= \frac{4}{3} \cdot \frac{6}{1} \cdot \frac{2}{3} \\
&= \frac{48}{9} \\
&= \frac{16}{3}
\end{aligned}$$

As such:

$$w : y = \frac{w}{y} = \frac{16}{3} = 16 : 3$$

Thus, **E** is the correct answer.

4. The acute angles of a right triangle are a° and b°, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

- A 2
- B 3
- C 5
- D 7
- E 11

Solution(s):

We know that the interior angles of a triangle add up to 180°, and since the triangle in question is a right triangle, by definition one of the interior angles must measure 90°. The remaining two acute angles, a° and b°, must therefore have a sum of $180^\circ - 90^\circ = 90^\circ$.

Let's begin by exploring the largest values of a° and going from there, as those will naturally yield the smallest values of b°.

The greatest possible value of a° is 89°. This makes $b^\circ = 90^\circ - 89^\circ = 1^\circ$, which is not prime, so this is not a valid possible case.

Moving on, the next largest possible value of a° is 83°. This makes $b^\circ = 90^\circ - 83^\circ = 7^\circ$, which is prime! Therefore, this is the smallest possible value of b.

Thus, **D** is the correct answer.

5. How many distinguishable arrangements are there of 1 brown tile, 1 purple tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable.)

A 210

B 420

C 630

D 840

E 1050

Solution(s):

We have $1+1+2+3=7$ total tiles, and as such, there are $7!$ total ways to order them. However, notice that this is overcounting, as it assumes that all tiles are indistinguishable, when in reality, the yellow and green tiles can be reordered amongst themselves without any change to the overall tiling pattern.

Let's remove this overcounting as follows. There are 3 yellow tiles, and as such, there are $3!=6$ ways to switch them around amongst themselves such that the overall tiling pattern is unchanged, as they are indistinguishable.

Similarly, there are 2 green tiles, and as such, there are $2!=2$ ways to switch them around amongst themselves such that the overall tiling pattern is unchanged, as they are indistinguishable.

Therefore, there are

$$\begin{aligned}\frac{7!}{6\cdot 2} &= 7\cdot 5\cdot 4\cdot 3\\ &= 7\cdot 60\\ &= 420\end{aligned}$$

possible tilings.

Thus, **B** is the correct answer.

6. Driving along a highway, Megan noticed that her odometer showed 15951 (miles). This number is a palindrome — it reads the same forward and backward. Then 2 hours later, the odometer displayed the next higher palindrome. What was her average speed, in miles per hour, during this 2-hour period?

- A 50
- B 55
- C 60
- D 65
- E 70

Solution(s):

We want to find the smallest palindrome larger than 15951, and to do so, we want to increase the value of the middle digit. This is because if we increased the value of any other digit with less place value (say, the ones place), we would also need to increase the value of the digit with greater place value (i.e. ten-thousands place).

Therefore, we must replace the number 9 to the next greatest integer: 10. As all digits must be between 0 and 9 (inclusive), we write a zero in place of the nine, and increase the thousands digit (and the tens — to preserve the palindrome).

As such, the next smallest palindrome larger than 15951 is 16061.

As Megan drives $16061 - 15951 = 110$ miles in 2 hours, she drives at an average of 55 mph.

Thus, **B** is the correct answer.

7. How many positive even multiples of 3 less than 2020 are perfect squares?

- A 7
- B 8
- C 9
- D 10
- E 12

Solution(s):

If a number is even, it must be a multiple of 2, and as such, if a number is a positive even multiple of 3, it is a multiple of 3 and 2, meaning that it is a multiple of 6.

As such, we can write arbitrary positive multiples of six as $6k$, for $k \geq 0$.

We want to find the number of positive even multiples of 3 — or more simply, positive integers of the form $6k$ — that are also perfect squares. Therefore, we are looking for integers less than 2020 of the form: $(6k)^2$.

Notice that for

$$k = 8, (6k)^2 = 2304 \geq 2020,$$

and for

$$k = 7, (6k)^2 = 1764 \leq 2020.$$

This means that the maximum k we can have is 7, and as $k > 0$ all $k : 1 \leq k \leq 7$ are valid solutions.

As such, there are 7 solutions.

Thus, **A** is the correct answer.

8. Points P and Q lie in a plane with $PQ = 8$. How many locations for point R in this plane are there such that the triangle with vertices P, Q, and R is a right triangle with area 12 square units?

A 2

B 4

C 6

D 8

E 12

Solution(s):

We know that the area of $\triangle PQR$, denoted $A(\triangle PQR)$, is equal to $\frac{1}{2} \cdot PQ \cdot h_R$, where h_R is equal to the height of $\triangle PQR$.

As $A(\triangle PQR) = 12$ and $PQ = 8$, we can see that $h_R = 3$.

WLOG, allow $P = (-4, 0)$ and $Q = (4, 0)$. With this in mind, let's case on the position of the right angle:

$\angle P = 90^\circ$. This implies that $R = (-4, \pm 3)$.

$\angle Q = 90^\circ$. This implies that $R = (4, \pm 3)$.

$\angle R = 90^\circ$. This implies that

$$\begin{aligned} 8^2 &= PR^2 + QR^2 \\ 8^2 &= (-4 - x)^2 + y^2 \\ &\quad + (4 - x)^2 + y^2 \\ 8^2 &= (-4 - x)^2 + (\pm 3)^2 \\ &\quad + (4 - x)^2 + (\pm 3)^2 \\ 8^2 &= (4 + x)^2 + (4 - x)^2 + 18 \\ 46 &= (4 + x)^2 + (4 - x)^2 \\ 46 &= 16 + 8x + x^2 \\ &\quad + 16 - 8x + x^2 \\ 14 &= 2x^2 \\ x &= \pm\sqrt{7} \end{aligned}$$

Thus, we have $R = (\pm\sqrt{7}, \pm 3)$.

As such, we have $2 + 2 + 4 = 8$ possible locations for R.

Thus, the correct answer is **D**.

9. How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

- A 1
- B 2
- C 3
- D 4
- E infinitely many

Solution(s):

We can rewrite the equation as

$$y^2 - 2y + x^{2020} = 0.$$

Using the quadratic formula, we know that:

$$y = \frac{2 \pm \sqrt{4 - 4(1)(x^{2020})}}{2}$$

As y must be an integer, and therefore a real number, we must have $4 - 4x^{2020} \geq 0$ to prevent negative square roots. This suggests:

$$\begin{aligned} 4 - 4x^{2020} &\geq 0 \\ x^{2020} &\leq 1 \\ -1 \leq x &\leq 1. \end{aligned}$$

Since x must be an integer for y to be an integer, this gives us three x values, each with either one or two corresponding y values. With this, we have the following four solutions:

$$(-1, 1)$$

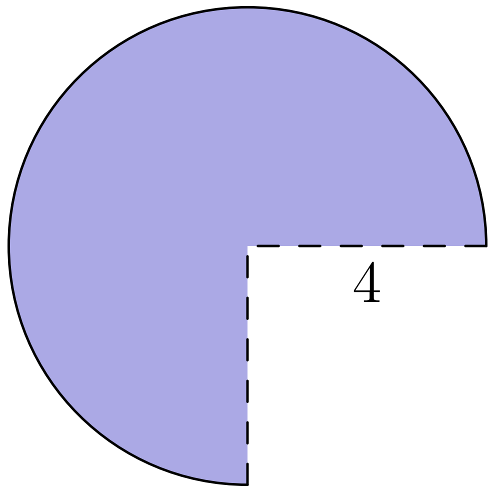
$$(0, 0)$$

$$(0, 2)$$

$$(1, 1)$$

Thus, **D** is the correct answer.

10. A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface area of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches?



A $3\pi\sqrt{5}$

B $4\pi\sqrt{3}$

C $3\pi\sqrt{7}$

D $6\pi\sqrt{3}$

E $6\pi\sqrt{7}$

Solution(s):

Remember that the volume of a cone is equal to $\frac{1}{3}\pi r^2 h$. Further notice that the circumference of the base of the cone is equal to the remaining circumference of the circle: $\frac{3}{4}$ the circumference of the complete circle.

Therefore, the circumference of the base of the cone is equal to:

$$\begin{aligned} C &= \frac{3}{4} \cdot 2(\pi)(4) \\ &= 6\pi \end{aligned}$$

This suggests that the radius of the cone's base (r') is equal to:

$$\begin{aligned} C &= 2\pi r' \\ 6\pi &= 2\pi r' \\ r' &= 3. \end{aligned}$$

Also, when we tape together the marked radii to form the cone, the radii in question become the slanted height of the cone. This can be used to find the actual height of the cone, which by the Pythagorean Theorem, is equal to:

$$\begin{aligned} SL^2 &= h^2 + (r')^2 \\ 4^2 &= h^2 + 3^2 \\ 16 &= h^2 + 9 \\ 7 &= h^2 \\ h &= \sqrt{7}. \end{aligned}$$

Thus, the volume of the cone is equal to:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi (r')^2 h \\ &= \frac{1}{3}\pi (3)^2 \cdot \sqrt{7} \\ &= \frac{1}{3}\pi 9\sqrt{7} \\ &= 3\pi\sqrt{7} \end{aligned}$$

Thus, the correct answer is **C**.

11. Ms. Carr asks her students to read any 5 of the 10 books on a reading list. Harold randomly selects 5 books from this list, and Betty does the same. What is the probability that there are exactly 2 books that they both select?

A $\frac{1}{8}$

B $\frac{5}{36}$

C $\frac{14}{45}$

D $\frac{25}{63}$

E $\frac{1}{2}$

Solution(s):

Assume that Harold has already picked his 5 books. Of these five books, there are $\binom{5}{2}$ ways that Betty can have picked exactly two of the same books as Harold, and $\binom{5}{3}$ ways that Betty can choose her other three books from the 5 books not on Harold's list.

As such, there are $\binom{5}{2}\binom{5}{3} = 100$ ways for Betty to choose her books such that she chooses exactly two books on Harold's list and three books not on Harold's list.

Therefore, as there are $\binom{10}{5} = 252$ ways that Betty can choose her books arbitrarily, and 100 of those choices satisfy the above conditions, the probability that they have exactly two books in common is:

$$\frac{100}{252} = \frac{25}{63}$$

Thus, **D** is the correct answer.

12. The decimal representation of

$$\frac{1}{20^{20}}$$

consists of a string of zeros after the decimal point, followed by a 9 and then several more digits. How many zeros are in that initial string of zeros after the decimal point?

- A 23
- B 24
- C 25
- D 26
- E 27

Solution(s):

Before tackling the problem itself, let's notice that a pattern emerges when dividing 1 by different integers, where a is one digit:

$$\begin{aligned}\frac{1}{a} &= 0.\underline{k}\cdots\\ \frac{1}{10a} &= 0.0\underline{k}\cdots\\ \frac{1}{10^2a} &= 0.00\underline{k}\cdots\\ &\vdots\end{aligned}$$

for some digit k (the dots represent the fact that there may be digits after k).

It seems that $1 \div (10^n a)$ will have n zeros after the decimal point.

Returning to the original problem, notice that $20^{20} = 10^{20} \cdot 2^{20}$. As $2^{10} = 1024 \approx 1000 = 10^3$, it follows that $2^{20} \approx 10^6$. Thus:

$$20^{20} \approx 10^6 \cdot 10^{20} = 10^{26}.$$

Therefore:

$$\frac{1}{20^{20}} \approx \frac{1}{10^{26}},$$

which, by the pattern established above, implies that there are 26 zeroes after the decimal point.

Thus, the correct answer is **D**.

13. Andy the Ant lives on a coordinate plane and is currently at $(-20, 20)$ facing east (that is, in the positive x-direction). Andy moves 1 unit and then turns 90° left. From there, Andy moves 2 units (north) and then turns 90° left. He then moves 3 units (west) and again turns 90° left. Andy continues his progress, increasing his distance each time by 1 unit and always turning left. What is the location of the point at which Andy makes the 2020th left turn?

A $(-1030, -994)$

B $(-1030, -990)$

C $(-1026, -994)$

D $(-1026, -990)$

E $(-1022, -994)$

Solution(s):

Let's begin by seeing how each step changes the Andy's coordinate:

Step 1: Andy moves one unit east and then turns left. Therefore, his new position is $(-19, 20)$, facing north.

Step 2: Andy moves two units north and then turns left. Therefore, his new position is $(-19, 22)$, facing west.

Step 3: Andy moves three units west and then turns left. Therefore, his new position is $(-22, 22)$, facing south.

Step 4: Andy moves four units south and then turns left. Therefore, his new position is $(-22, 18)$, facing east.

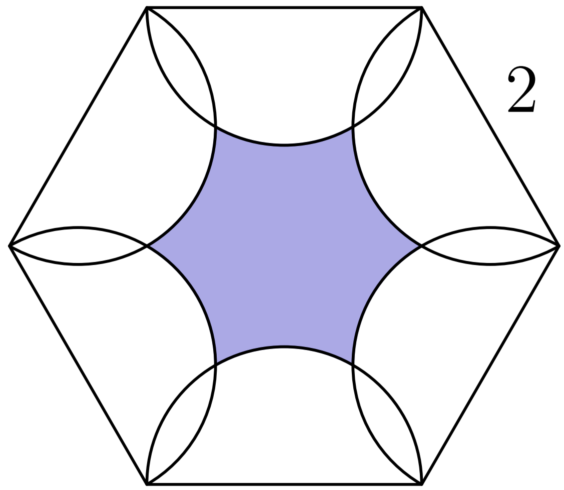
A cyclic pattern seems to emerge, as every four steps Andy takes, it seems that if his original position is (a, b) facing east, is new position is $(a-2, b-2)$ facing east.

As there are 505 such 4-step cycles in the 2020 steps Andy takes, it follows that Andy's final position is:

$$\begin{aligned} &(-20-505(2), 20-505(2)) \\ =&(-20-1010, 20-1010) \\ =&(-1030, -990) \end{aligned}$$

Thus, **B** is the correct answer.

14. As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded region — inside the hexagon but outside all of the semicircles?



A $6\sqrt{3} - 3\pi$

B $\frac{9\sqrt{3}}{2} - 2\pi$

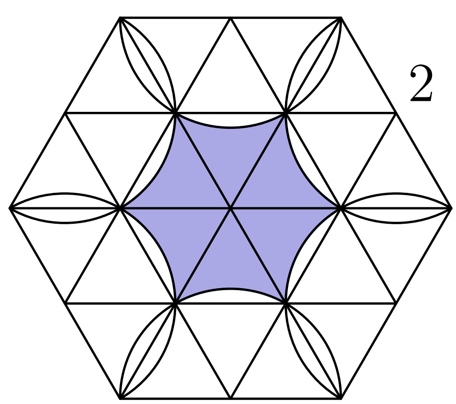
C $\frac{3\sqrt{3}}{2} - \frac{\pi}{3}$

D $3\sqrt{3} - \pi$

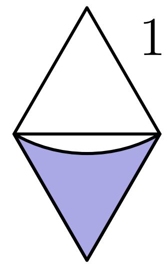
E $\frac{9\sqrt{3}}{2} - \pi$

Solution(s):

Firstly construct the following lines between points on the hexagon:



Observe that the area of the shaded region is simply 6 times the area of the following shaded region:



As the entire figure is simply two equilateral triangles with side length 1, the total area is:

$$2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

The unshaded region has an area equal to a circular sector with radius 1 and an angle $\frac{\pi}{3}$ (as the triangle is equilateral). As such, the area of the unshaded region is:

$$\frac{\frac{\pi}{3}}{2\pi} \cdot \pi = \frac{1}{6} \cdot \pi = \frac{\pi}{6}$$

Meaning the area of the shaded region is:

$$\frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

And as such, the area of the larger, complete shaded area in the hexagon is:

$$6 \cdot \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) = 3\sqrt{3} - \pi$$

Thus, **D** is the correct answer.

15. Steve wrote the digits 1, 2, 3, 4, and 5 in order repeatedly from left to right, forming a list of $10,000$ digits, beginning $123451234512\ldots$

He then erased every third digit from his list (that is, the 3rd, 6th, 9th, $\ldots$ digits from the left), then erased every fourth digit from the resulting list (that is, the 4 th, 8th, 12th, $\ldots$ digits from the left in what remained), and then erased every fifth digit from what remained at that point.

What is the sum of the three digits that were then in the positions $2019, 2020, 2021$?

- A 7
- B 9
- C 10
- D 11
- E 12

Solution(s):

Let's begin by analyzing the list of digits after the removal of the third digits.

Although the list initially cycles with a period of 5 (as in, the list repeats itself every 5 digits), as $\text{lcm}(3,5) = 15$, the relative positions from which we delete digits from repeats every 15 digits. Therefore:

$$123451234512345$$

$$12\not{3}45\not{1}23\not{4}51\not{2}34\not{5}$$

Therefore, the new list is:

$$1245235134$$

and repeats every 10 digits.

Similarly, we consider the same procedure for removing the fourth digits.

As $\text{lcm}(4,10) = 20$, let's consider the first 20 digits of this list when making our deletions, as after this point, the relative positions of our deletions will repeat.

$$12452351341245235134$$

$$124\not{5}235\not{1}341\not{2}452\not{3}513\not{4}$$

Therefore, the new list is:

$$124235341452513$$

and repeats every 15 digits.

Lastly, we consider the same procedure for removing the fifth digits.

As $\text{lcm}(5, 15) = 15$, let's consider the first 15 digits of this list when making our deletions, as after this point, the relative positions of our deletions will repeat.

$$124235341452513$$

$$1242\not{3}5341\not{4}5251\not{3}$$

Therefore, the new list is:

$$124253415251$$

and repeats every 12 digits.

As $2019 \equiv 3 \bmod 12$, we know that the 2019th digit is equal to the 3rd digit in the list above. As such, it follows that:

$$\text{position } 2019 = 4$$

$$\text{position } 2020 = 2$$

$$\text{position } 2021 = 5$$

Therefore, the sum of these values is $4 + 2 + 5 = 11$.

Thus, the correct answer is **D**.

16. Bela and Jenn play the following game on the closed interval $[0, n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0, n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game?

A Bela will always win.

B Jenn will always win.

C Bela will win if and only if n is odd.

D Jenn will win if and only if n is odd.

E Bela will win if and only if n >8.

Solution(s):

Consider the following strategy:

Bela begins by choosing the middle value $\frac{n}{2}$ in $[0, n]$. From then on, whatever value say — k — Jenn chooses, Bela will always choose $n - k$. As Bela goes first, and chooses the middle value, she will always be able to choose the mirrored value of Jenn's move and Jenn will run out of number to pick first.

Therefore, following this strategy, Bela will always win.

Thus, **A** is the correct answer.

17. There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9 people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?

A 11

B 12

C 13

D 14

E 15

Solution(s):

We case upon the number of opposite pairs. As in, we count the number of possible pairs when we have $0, 1, 2, \cdots, 5$ pairs whose members are across the circle from one another. Then, we will add up all these values to find the total number of ways to split up the 10 people.

0 **opposite pairs**

In this case, we select pairs by either matching person 1 with person 2, and person 3 with person 4, and so on. Or, we can match person 1 with person 10, and person 9 with person 8, and so on. As such, there are 2 ways to pair up individuals in this case.

1 **opposite pair**

In this case, we have 5 possible pairs where a person is matched with the person across from them, and everyone else pairs with the person next to them.

2 **opposite pairs**

We cannot have 2 opposite pairs, as there would be one person on either side that will not have a pair adjacent to them.

3 **opposite pairs**

In this case, we have $\binom{5}{3} = 10$ possible ways to match a person with the person across from them, where everyone else pairs with the person next to them. However, notice that there are duplicates, so we divide the 10 possibilities by 2 to get 5.

4 **opposite pairs**

We cannot have 4 opposite pairs, as there would be one person on either side that will not have a pair adjacent to them.

5 **opposite pairs**

In this case, there is only one way to do this.

As such, there are $2 + 5 + 5 + 1 = 13$ ways to pair up individuals such that the members of each pair know each other.

Thus, the correct answer is **C**.

18. An urn contains one red ball and one blue ball. A box of extra red and blue balls lie nearby. George performs the following operation four times: he draws a ball from the urn at random and then takes a ball of the same color from the box and returns those two matching balls to the urn. After the four iterations the urn contains six balls. What is the probability that the urn contains three balls of each color?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution(s):

Begin by noting that there are $\binom{4}{2} = 6$ ways to arrange two red and two blue balls in different orders.

Now, given an arbitrary arrangement of these four balls, we want to find the probability of said arrangement happening. We will show that the probability of each arrangement is the same.

We first want to find the total number of choices we have to choose from, and as there are 2 choices for the first step, 3 choices for the second step, 4 choices for the third step, and 5 choices for the fourth step, it follows that there are

$$2 \cdot 3 \cdot 4 \cdot 5 = 120$$

different ways to choose an arrangement.

Regardless of the details of the specific arrangement we aim to get, the first time a red ball is added will come from 1 choice, and the second time will come from 2 choices, as that is the initial number of red balls in the urn. This also holds for the greens. As such, the number of successful choices is:

$$(1 \cdot 2)^2 = 4$$

As such, for an arbitrary ordering of two red and two blue balls, we have a $\frac{4}{120} = \frac{1}{30}$ probability. As there are 6 possible arrangements, the total probability is $\frac{1}{5}$.

Thus, the correct answer is **B**.

19. In a certain card game, a player is dealt a hand of 10 cards from a deck of 52 distinct cards. The number of distinct (unordered) hands that can be dealt to the player can be written as $158A00A4AA0$. What is the digit A?

A 2

B 3

C 4

D 6

E 7

Solution(s):

The number of distinct hands that can be dealt to the player is equal to:

$$\binom{52}{10} = \frac{52!}{42! \cdot 10!} = 10 \cdot 17 \cdot 13 \cdot 7 \cdot 47 \cdot 46 \cdot 11 \cdot 43 = 158A00A4AA0$$

Therefore:

$$17 \cdot 13 \cdot 7 \cdot 47 \cdot 46 \cdot 11 \cdot 43 = 158A00A4AA$$

To find the units digit, we can find the value of this expression mod 10:

$$\begin{aligned} A &\equiv 17 \cdot 13 \cdot 7 \cdot 47 \cdot 46 \cdot 11 \cdot 43 \bmod 10 \\ &\equiv 7 \cdot 3 \cdot 7 \cdot 7 \cdot 6 \cdot 1 \cdot 3 \bmod 10 \\ &\equiv 1 \cdot 9 \cdot 6 \cdot 3 \bmod 10 \\ &\equiv 9 \cdot 8 \bmod 10 \\ &\equiv 2 \bmod 10 \end{aligned}$$

Therefore, as $0 \leq A \leq 9$, and $A \equiv 2 \bmod 10$, it follows that $A = 2$.

Thus, **A** is the correct answer.

20. Let B be a right rectangular prism (box) with edges lengths 1, 3, and 4, together with its interior. For real $r \geq 0$, let $S(r)$ be the set of points in 3-dimensional space that lie within a distance r of some point in B. The volume of $S(r)$ can be expressed as

$$ar^3 + br^2 + cr + d,$$

where a, b, c, and d are positive real numbers. What is $\frac{bc}{ad}$?

- A 6
- B 19
- C 24
- D 26
- E 38

Solution(s):

Consider $S(r)$ at the following regions:

The rectangular prism itself (B) :

In this region, $r = 0$, and therefore the volume of $S(0) = d = 1 \cdot 3 \cdot 4 = 12.$

The extensions of the faces of B :

The volumes of this region are equal to the sums of the volumes of the boxes produced by extending every face on B to a distance r. Therefore, the total volume of this region is:

$$2(1 \cdot 3 + 1 \cdot 4 + 3 \cdot 4) \cdot r = 38r$$

The quarter-cylinders at each of the edges of B :

There are 12 edges on B, meaning that there are 12 quarter-cylinders. 4 of which have length 4,4 of which have length 3, and 4 of which have length 2. All of these cylinders have radius r. Therefore, the volumes is:

$$\frac{1}{4} \cdot 4\pi r^2(4 + 3 + 1) = 8\pi r^2$$

The eighth-spheres at every vertex of B :

There are 8 vertices on B, and the eighth-sphere corresponding to each one has a radius r. As such, the total volume is:

$$8 \cdot \frac{1}{8} \cdot \left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi r^3$$

Therefore, the total area enclosed by $S(r) = \frac{4}{3}\pi r^3 + 8\pi r^2 + 38r + 12$

As such:

$$a = \frac{4}{3}\pi$$

$$b = 8\pi$$

$$c = 38$$

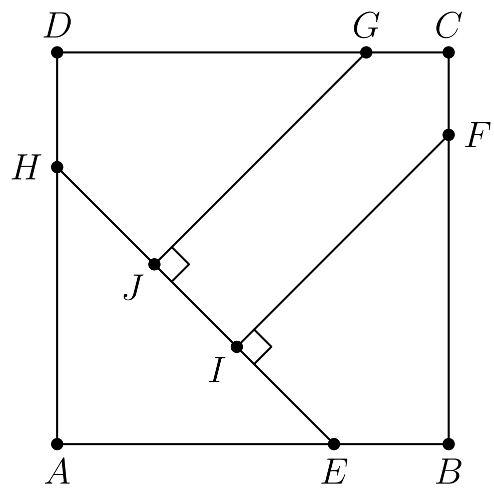
$$d = 12$$

Therefore:

$$\frac{bc}{ad} = \frac{8 \cdot 38\pi}{16\pi} = 19$$

Thus, the correct answer is **B**.

21. In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE = AH$. Points F and G lie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area 1. What is FI^2?



A $\frac{7}{3}$

B $8 - 4\sqrt{2}$

C $1 + \sqrt{2}$

D $\frac{7}{4}\sqrt{2}$

E $2\sqrt{2}$

Solution(s):

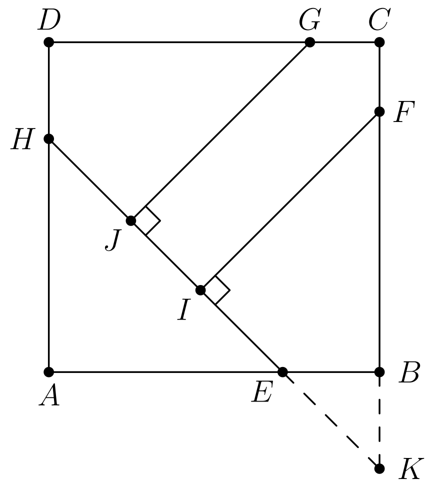
As we know that the sum of the areas of

$$\begin{aligned} &AEH + BFIE + \\ &DHJG + FCGJI \\ &= 1 + 1 + 1 + 1 \\ &= 4, \end{aligned}$$

we know that the side length of the square is 2.

Similarly, as AEH has an area of 1, we know that $AH = AE = \sqrt{2}$.

Now, let us extend the figure as follows:



With the construction of K as the eventual intersection of FB and IE, we can notice that BEK is a right isosceles triangle, with side length $2-\sqrt{2}$. Therefore, its area is $3-2\sqrt{2}$.

From this, we can further deduce that FIK is also a right isosceles triangle, with area

$$\begin{aligned}\frac{1}{2}FI \cdot IK &= \frac{1}{2}FI^2 \\ &= 1+3-2\sqrt{2} \\ &= 4-2\sqrt{2},\end{aligned}$$

implying that

$$FI^2 = 8-4\sqrt{2}.$$

Thus, the correct answer is **B**.

22. What is the remainder when $2^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$?

A 100

B 101

C 200

D 201

E 202

Solution(s):

Observe that:

$$\begin{aligned} 2^{202} + 202 &= \left(2^{101}\right)^2 + 2 \cdot 2^{101} + 1 \\ &\quad - 2 \cdot 2^{101} + 201 \\ &= \left(2^{101} + 1\right)^2 \\ &\quad - 2 \cdot 2^{101} + 201 \\ &= a^2 - b^2 + 201, \end{aligned}$$

where $a = 2^{101} + 1$ and $b = 2^{51}$. Thus, by difference of squares:

$$\begin{gathered} a^2 - b^2 + 201 \\ = (a+b)(a-b) + 201 \end{gathered}$$

Which means that:

$$\begin{gathered} 2^{202} + 202 = \\ (2^{101} + 2^{51} + 1)(2^{101} - 2^{51} + 1) \\ +201. \end{gathered}$$

Therefore,

$$\begin{gathered} 2^{202} + 202 \equiv \\ (2^{101} + 2^{51} + 1)(2^{101} - 2^{51} + 1) + \\ 201 \equiv 201 \bmod (2^{101} + 2^{51} + 1). \end{gathered}$$

Thus, the correct answer is **D**.

23. Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1), C(-1,-1)$, and $D(1,-1)$. Consider the following four transformations:

- L, a rotation of 90° counterclockwise around the origin;
- R, a rotation of 90° clockwise around the origin;
- H, a reflection across the x-axis; and
- V, a reflection across the y-axis.

Each of these transformations maps the squares onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.)

A 2^{37}

B $3 \cdot 2^{36}$

C 2^{38}

D $3 \cdot 2^{37}$

E 2^{39}

Solution(s):

Notice that all of these transformations result in each vertex being moved to some adjacent vertex. For example, after one transformation, vertex A will always be at either $(-1,1)$ or $(1,-1)$.

It follows that after any odd number of transformations, vertex A will always be either in quadrant II or IV, and after any even number of transformations, A will always be in either quadrant I or III. This rule holds for all other vertices, and after 19 moves, we are left with the following four possible states:

- $\begin{pmatrix} A & B \\ D & C \end{pmatrix}$

- $\begin{pmatrix} C & B \\ D & A \end{pmatrix}$
- $\begin{pmatrix} A & D \\ B & C \end{pmatrix}$
- $\begin{pmatrix} C & D \\ B & A \end{pmatrix}$

From each of these positions, it is possible to return to the original position with only one transformation as transformation 20: In order, they are V, L, R, H.

As such, as there are 4^{19} ways to make the first 19 transformations, and all of these transformations lead to one of four final configurations, each with one valid move to get back to the original position, we conclude that there are exactly

$$4^{19} = 2^{38}$$

sequences of 20 transformations that map the square back to its original position.

Thus, **C** is the correct answer.

24. How many positive integers n satisfy

$$\frac{n+1000}{70} = \lfloor \sqrt{n} \rfloor?$$

(Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

A 2

B 4

C 6

D 30

E 32

Solution(s):

We know that by definition that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Therefore:

$$\frac{n+1000}{70} \leq \sqrt{n} < \frac{n+1070}{70}$$

Solving each inequality separately:

$$\begin{aligned} \frac{n+1000}{70} &\leq \sqrt{n} \\ n+1000 &\leq 70\sqrt{n} \\ n - 70\sqrt{n} + 1000 &\leq 0 \\ (\sqrt{n}-50)(\sqrt{n}-20) &\leq 0 \end{aligned}$$

$$\begin{gathered} 20 \leq \sqrt{n} \leq 50 \\ 400 \leq n \leq 2500 \end{gathered}$$

and

$$\begin{gathered} \sqrt{n} < \frac{n+1070}{70} \\ 0 < n - 70\sqrt{n} + 1070. \end{gathered}$$

Using the qudratic formula yields

$$\sqrt{n} < 35 - \sqrt{155}$$

$$0 \leq n < 1380 - 70\sqrt{155}$$

or

$$\sqrt{n} > 35 + \sqrt{155}$$

$$1380 + 70\sqrt{155} < n.$$

Therefore, as we now know that n is in the interval $[400, 1380 - 70\sqrt{155})$ or in $(1380 + 70\sqrt{155}, 2500]$.

Furthermore, we know that n is an integer, and as $\dfrac{n + 1000}{70} = \lfloor\sqrt{n}\rfloor$ must be an integer, then it follows that:

$$n + 1000 \equiv 0 \bmod 70$$

$$n + 20 \equiv 0 \bmod 70$$

$$n \equiv -20 \bmod 70$$

$$n \equiv 50 \bmod 70$$

As such, $n = 70k + 50$, for some integer k. Therefore, we substitute to see that:

$$k \in [5, 19 - \sqrt{155})$$

$$\implies k \in [5, 6]$$

or

$$k \in (19 + \sqrt{155}, 35]$$

$$\implies k \in [32, 35].$$

And so, we have 6 values of k, and as each k corresponds to exactly one n, we have 6 solutions for n.

Thus, **C** is the correct answer.

25. Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n = f_1 \cdot f_2 \cdots f_k,$$

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

A 112

B 128

C 144

D 172

E 184

Solution(s):

Notice that $96 = 2^5 \cdot 3$. To find all the ways to factor 96 with the conditions listed above, we must find all the ways to group our five $2'$s and one 3 such that each group has at least one element, from which we multiply all the elements in each group to get our factor pair.

This can be solved using a method similar to stars and bars, as we can depict this situation as:

$$(2, 2, |, 2, 2, 2, |, 3) = 4 \cdot 8 \cdot 3$$

Where we have k groups created by $k - 1$ dividers, and as we have 6 ways to arrange the prime factors and 2^5 ways to place the dividers such that each group has at least one element, we have

$$6 \cdot 2^5$$

total ways to arrange factors.

However, we must notice that whenever we have a group that includes 3 and at least one other prime factor (i.e. any groups that have a product of $6, 12, 24, 48, 96$) such that the factor that this group represents is of the form $3 \cdot$

2^i, then we will overcount this case by a factor of $i+1$. In other words, we double count our $6'$s, triple count our $12'$s, and quadruple count our $24'$s.

This is because we consider different arrangements within a group to constitute entirely different groups, when in fact, they are the same. For example:

$$(2, 2, |, 2, 3, |, 2, 2) = 4 \cdot 6 \cdot 4$$

$$(2, 2, |, 3, 2, |, 2, 2) = 4 \cdot 6 \cdot 4$$

both represent the same factor triple, but are considered to be distinct.

We must compensate for this overcounting by counting the number of these distinct but equal cases and subtracting. To do this, consider all cases where 3 is grouped with at least one 2. This can be similarly represented as the previous case, only this time with four $2'$s and one 6 (as $6 = 2 \cdot 3$), resulting in groups such as:

$$(2, 2, 2, |, 2, 6)$$

With this representation, there are 5 ways to arrange the elements, and 2^4 ways to place dividers, and as such, there are $5 \cdot 2^4$ overcounted cases.

Therefore, all that is left is to subtract our overcount, and from this, we see that

$$\begin{aligned} D(96) &= 6 \cdot 2^5 - 5 \cdot 2^4 \\ &= 2^4(12 - 5) \\ &= 7 \cdot 16 \\ &= 112 \end{aligned}$$

Thus, **A** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2020B