2021 AMC 10B Spring Solutions

Typeset by: LIVE, by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2021B/solutions

Problems © Mathematical Association of America. Reproduced with permission.

1. How many integer values of x satisfy $|x| < 3\pi$?

A 9

B 10

C 18

D 19

E 20

Solution(s):

Every integer from -9 to 9, inclusive, works. This yields $9 - (-9) + 1 = 19$ solutions.

Thus, the correct answer is **D**.

2. What is the value of

$$\sqrt{\left(3-2\sqrt{3}\right)^2}+\sqrt{\left(3+2\sqrt{3}\right)^2}?$$

A 0

B $4\sqrt{3}-6$

C 6

D $4\sqrt{3}$

E $4\sqrt{3}+6$

Solution(s):

We know

$$\sqrt{\left(3-2\sqrt{3}\right)^2}+\sqrt{\left(3+2\sqrt{3}\right)^2}$$
$$=|3-2\sqrt{3}|+|3+2\sqrt{3}|.$$

Since $3<2\sqrt{3}$, we know that $3-2\sqrt{3}<0$.

Therefore, our desired equation expression is equal to

$$-3+2\sqrt{3}+3+2\sqrt{3}$$
$$=4\sqrt{3}.$$

Thus, the correct answer is **D**.

3. In an after-school program for juniors and seniors, there is a debate team with an equal number of students from each class on the team. Among the 28 students in the program, 25% of the juniors and 10% of the seniors are on the debate team. How many juniors are in the program?

A 5

B 6

C 8

D 11

E 20

Solution(s):

Let the number of juniors be j and the number of seniors be s. Then, $j + s = 28$ and $0.25j = 0.1s$. This means $2.5j = s$, so $3.5j = 28$. This makes $j = 8$.

Thus, the correct answer is **C**.

4. At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?

A 23

B 32

C 37

D 41

E 64

Solution(s):

There are $2 \cdot 23 = 46$ students with blue shirts that are in a pair with just blue shirts. This means there are $57 - 46 = 11$ students in blue shirts who are paired with someone wearing a yellow shirts, meaning exactly 11 people wearing yellow shirts are paired with someone wearing a blue shirt.

This leaves just 64 students wearing a yellow shirt who is working with someone else wearing a yellow shirt. This yields $\dfrac{64}{2} = 32$ pairs.

Thus, the answer is **B**.

5. The ages of Jonie's four cousins are distinct single-digit positive integers. Two of the cousins' ages multiplied together give 24, while the other two multiply to 30. What is the sum of the ages of Jonie's four cousins?

- A 21
- B 22
- C 23
- D 24
- E 25

Solution(s):

Since the last two are multiplied to 30 and both are single-digit numbers, one of them must be 5, making the other person 6. The first two are of ages that multiply to 24. The only pair of single-digit numbers whos product is 24 and none of them are 4 or 6 is the pair $3, 8$. Thus, the ages are $3, 5, 6, 8$, making their sum 22.

Thus, the answer is **B**.

6. Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

- A 74
- B 75
- C 76
- D 77
- E 78

Solution(s):

Let the number of people in the first class be $3x$. This means the number of people in the second class is $4x$.

Thus, the sum of the scores of the first class is $84 \cdot 3x = 252x$ and the sum of the scores for the people in the second class is $70 \cdot 4x = 280x$. This means the total sum is $532x$, with $7x$ people.

Therefore, the average of all the students is $\dfrac{532x}{7x} = 76$.

Thus, the correct answer is **C**.

7. In a plane, four circles with radii $1, 3, 5,$ and 7 are tangent to line ℓ at the same point A, but they may be on either side of ℓ. Region S consists of all the points that lie inside exactly one of the four circles. What is the maximum possible area of region S?

A 24π

B 32π

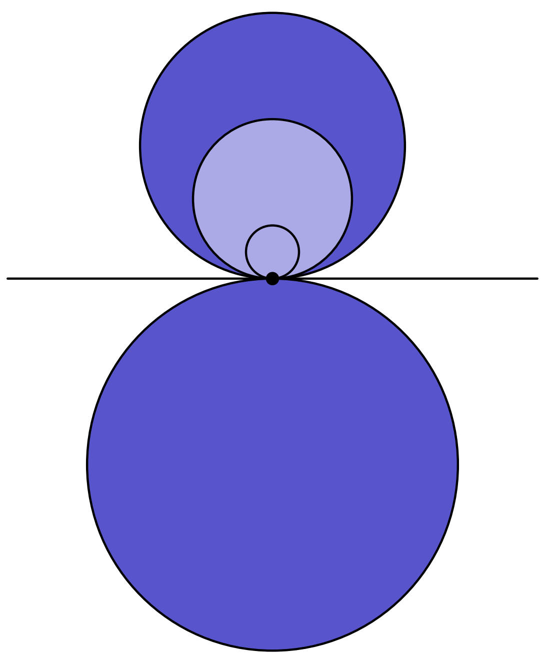
C 64π

D 65π

E 84π

Solution(s):

Consider the following diagram where the lighter colored area makes up region S:



The circles can be in only two locations. We first place the largest circle and then the second largest circle in the opposite location. After this, the circle of radius 3 must be placed on one of the two sides. To minimize the area lost with the last circle, we put it inside the third circle.

This yields an area of $7^2\pi + 5^2\pi - 3^2\pi = 65\pi$.

Thus, the answer is **D**.

8. Mr. Zhou places all the integers from 1 to 225 into a 15 by 15 grid. He places 1 in the middle square (eighth row and eighth column) and places other numbers one by one clockwise, as shown in part in the diagram below. What is the sum of the greatest number and the least number that appear in the second row from the top?

$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	21	22	23	24	25	$\cdots$
$\cdots$	20	7	8	9	10	$\cdots$
$\cdots$	19	6	1	2	11	$\cdots$
$\cdots$	18	5	4	3	12	$\cdots$
$\cdots$	17	16	15	14	13	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

A 367

B 368

C 369

D 379

E 380

Solution(s):

The number on the top left corner is 211, so the number under it would be 210. This is the greatest number on the second row from the top since 211 to 225 are the largest numbers in the square and they are in the top row. now, we must find the least number in the row. Besides the left most number, the squares are increasing by 1 from left to right.

Therefore, the second square is the least square. Note that the diagonal from the center to the upper right corner has all the odd squares, so the 14th number in the row is 169. Thus, the second number in the row is $169 - 12 = 157$, so that is the least number in the row. The sum is $157 + 210 = 367$.

Thus, the answer is **A**.

9. The point $P(a, b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1, 5)$ and then reflected about the line $y = -x$. The image of P after these two transformations is at $(-6, 3)$. What is $b - a$?

A 1

B 3

C 5

D 7

E 9

Solution(s):

To undo these steps, we first reflect $(-6, 3)$ back across $y = -x$. This would be $(-3, 6)$.

Next, we need to undo the rotation, so we rotate $(-3, 6)$ around $(1, 5)$ by 90° clockwise.

We can do this by translating our point , rotating around the origin, and translate back. The first translation would be $(-4, 1)$. Then rotating would be $(1, 4)$. Then, translating back would be $2, 9$. This would make $b = 9, a = 2$, so $b - a = 7$.

Thus, the answer is **D**.

10. An inverted cone with base radius $12\mathrm{cm}$ and height $18\mathrm{cm}$ is full of water. The water is poured into a tall cylinder whose horizontal base has radius of $24\mathrm{cm}$. What is the height in centimeters of the water in the cylinder?

A 1.5

B 3

C 4

D 4.5

E 6

Solution(s):

The volumes must be the same since the water is poured from one to another. The area of the cone is $\frac{r^2h\pi}{3} = \frac{12^2 \cdot 18\pi}{3} = 864\pi$.

The volume of the cylinder is $r^2h\pi = 24^2h\pi = 576h\pi$. This makes $576h\pi = 864\pi$, so $h = 1.5$.

Thus, the answer is **A**.

11. Grandma has just finished baking a large rectangular pan of brownies. She is planning to make rectangular pieces of equal size and shape, with straight cuts parallel to the sides of the pan. Each cut must be made entirely across the pan. Grandma wants to make the same number of interior pieces as pieces along the perimeter of the pan. What is the greatest possible number of brownies she can produce?

A 24

B 30

C 48

D 60

E 64

Solution(s):

The number of interior pieces is $(l-2)(w-2)$. This would be half of the total number of pieces, which is lw. This means $\frac{lw}{2}=(l-2)(w-2)$.

Multiplying out yields $lw-4l-4w+8=0$, so $(l-4)(w-4)=8$. This yields that $(l,w)=(5,12),(6,8)$, or some permutation of it. Thus, $lw=48$ or 60.

Therefore, the largest number of brownies is 60.

Thus, the answer is **D**.

12. Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

A $1:16$

B $1:15$

C $1:14$

D $1:8$

E $1:3$

Solution(s):

Using prime factorization, we get

$$N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2.$$

If we have an odd divisor x of N, then $2x, 4x, 8x$ are divisors of N, which has a combined sum of $14x$. If we take the sum of every odd divisor, then the even divisors must have a sum which is 14 times of the sum of the odd divisors.

Thus, the answer is **C**.

13. Let n be a positive integer and d be a digit such that the value of the numeral $\underline{32d}$ in base n equals 263, and the value of the numeral $\underline{324}$ in base n equals the value of the numeral $\underline{11d1}$ in base six. What is $n + d$?

A 10

B 11

C 13

D 15

E 16

Solution(s):

The first statement means

$$3n^2 + 2n + d = 263.$$

Similarly, the second statement means

$$3n^2 + 2n + 4$$

$$= 6^3 + 6^2 + 6d + 1$$

$$= 253 + 6d.$$

Subtracting these shows us that

$$4 - d = 6d - 10$$

$$7d = 14$$

$$d = 2.$$

Therefore, $3n^2 + 2n + 2 = 263$, so $n(3n + 2) = 261$.

This implies $n = 9$. Therefore, $n + d = 11$.

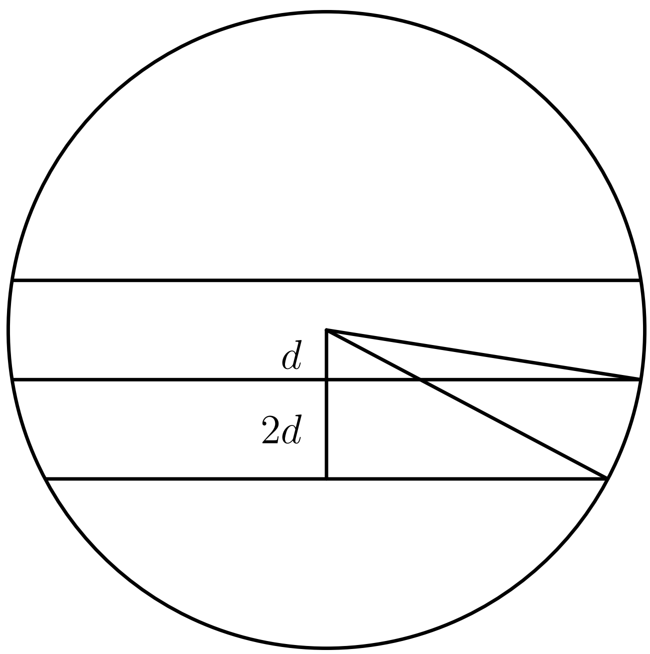
Thus, the answer is **B**.

14. Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38,$ and 34. What is the distance between two adjacent parallel lines?

- A $5\frac{1}{2}$
- B 6
- C $6\frac{1}{2}$
- D 7
- E $7\frac{1}{2}$

Solution(s):

Consider the following diagram:



Let the radius be r. Then, we can make the distance between two chords be $2d$. The two chords of length 38 are equidistant from the center, so the distance from the center to the other chord is $3d$. Thus, $r^2 + 19^2 = (3r)^2 + 17^2 \implies 19^2 - 17^2 = 8d^2$, so $8d^2 = 72$. Therefore, $2d = 6$.

Thus, the answer is **B**.

15. The real number x satisfies the equation

$$x + \frac{1}{x} = \sqrt{5}.$$

What is the value of $x^{11} - 7x^7 + x^3$?

A -1

B 0

C 1

D 2

E $\sqrt{5}$

Solution(s):

Since $x + \frac{1}{x} = \sqrt{5}$, squaring yields

$$x^2 + 2 + \frac{1}{x^2} = 5$$

$$x^2 + \frac{1}{x^2} = 3.$$

Squaring again yields

$$x^4 + 2 + \frac{1}{x^4} = 9$$

$$x^4 - 7 + \frac{1}{x^4} = 0.$$

Multiplying by x^7 yields

$$x^{11} - 7x^7 + x^3 = 0.$$

Thus, the answer is **B**.

16. Call a positive integer an uphill integer if every digit is strictly greater than the previous digit. For example, $1357, 89$, and 5 are all uphill integers, but $32, 1240$, and 466 are not. How many uphill integers are divisible by 15?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution(s):

If a number is divisible by 15, it has a units digit of 0 or 5. If the units digit is 0 and the digits are strictly increasing, then the number is 0, which isn't positive. Therefore, we can just look at numbers with a units digit of 5.

Next, we need to find uphill integers that are a multiple of 3. This means the other digits are a subset of $\{1, 2, 3, 4\}$. Taking the sum of the set must have a remainder of 1 when divided by 3. Also, having or taking out 3 wouldn't affect the remainder, so we can take the number of subsets without a 3 and multiply it by 2. There are only 3 such subsets, namely $\{1\}, \{4\}$, and $\{1, 2, 4\}$. Thus, there are 6 total subsets.

Thus, the correct answer is **C**.

17. Ravon, Oscar, Aditi, Tyrone, and Kim play a card game. Each person is given 2 cards out of a set of 10 cards numbered $1, 2, 3, \ldots, 10$. The score of a player is the sum of the numbers of their cards. The scores of the players are as follows: Ravon--11, Oscar--4, Aditi--7, Tyrone--16, Kim--17. Which of the following statements is true?

- A Ravon was given card 3.
- B Aditi was given card 3.
- C Ravon was given card 4.
- D Aditi was given card 4.
- E Tyrone was given card 7.

Solution(s):

If there are 2 cards for Oscar that add up to 4, he must have both 1 and 3. This eliminates choices A and B.

If there are 2 cards for Aditi that add up to 7, he must have both 2 and 5 so she doesn't have 1 and 3.

If someone has 4, their sum must be equal to or under 14 since the other number must be under or equal to 10. Thus, Ravon must have the 4 and 7, making C true and D and E false.

Thus, the answer is **C**.

18. A fair 6-sided die is repeatedly rolled until an odd number appears. What is the probability that every even number appears at least once before the first occurrence of an odd number?

A $\dfrac{1}{120}$

B $\dfrac{1}{32}$

C $\dfrac{1}{20}$

D $\dfrac{3}{20}$

E $\dfrac{1}{6}$

Solution(s):

The probability that the first number is even is $\frac{3}{6}$.

The probability that the second distinct number is even is $\frac{2}{5}$.

The probability that the third distinct number is even is $\frac{1}{4}$.

The combined probability is

$$\frac{3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4} = \frac{1}{20}.$$

Thus, the answer is **C**.

19. Suppose that S is a finite set of positive integers.

If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S.

What is the average value of all the integers in the set S?

A 36.2

B 36.4

C 36.6

D 36.8

E 37

Solution(s):

Let the sum of all the integers be s, the greatest number be g, the least number be l, and the size of S be n.

From the info given, we know

$$\frac{s-l}{n-1} = 40$$

$$\frac{s-g}{n-1} = 32.$$

Subtracting these yields

$$\frac{g-l}{n-1} = 8.$$

Since we know $g - l = 72$, we know $\frac{72}{n-1} = 8$, so $n = 10$.

We also know

$$\frac{s-g-l}{n-2} = 35,$$

so

$$s - g - l = 8 \cdot 35$$

$$= 280.$$

Since

$$\frac{s-l}{n-1} = 40,$$

we know$s - l = 360$. This makes $g = 80$. Using $g = l + 72$, we get $l = 8$. Thus, $s = 368$.

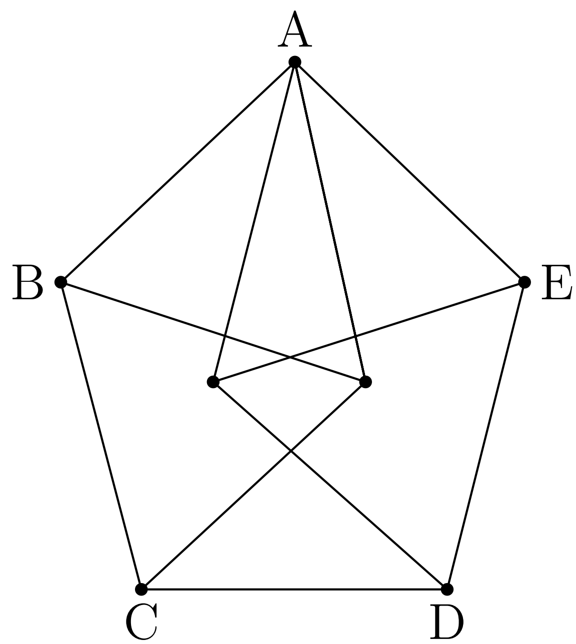
The average is

$$\frac{s}{n} = \frac{368}{10}$$

$$= 36.8.$$

Thus, the answer is **D**.

20. The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m}+\sqrt{n}$, where m and n are positive integers. What is $m+n$?



A 20

B 21

C 22

D 23

E 24

Solution(s):

First, we can find the length of AC and AD. Since the altitude of an equilateral triangle of side length 2 is $\sqrt{3}$, we know

$$AC = AD = 2\sqrt{3} = \sqrt{12}.$$

Note that the area of an equilateral triangle with side length 2 has area $\sqrt{3}$. This means that ABC and AED combined have an area of $2\sqrt{3} = \sqrt{12}$ since they are 4 halves of an equilateral triangle.

The rest of the area is ACD. Since $AC = AD = \sqrt{2}$ and $CD = 2$, the altitude from A to CD is

$$\sqrt{(\sqrt{12})^2 - \left(\frac{2}{2}\right)^2} = \sqrt{11}$$

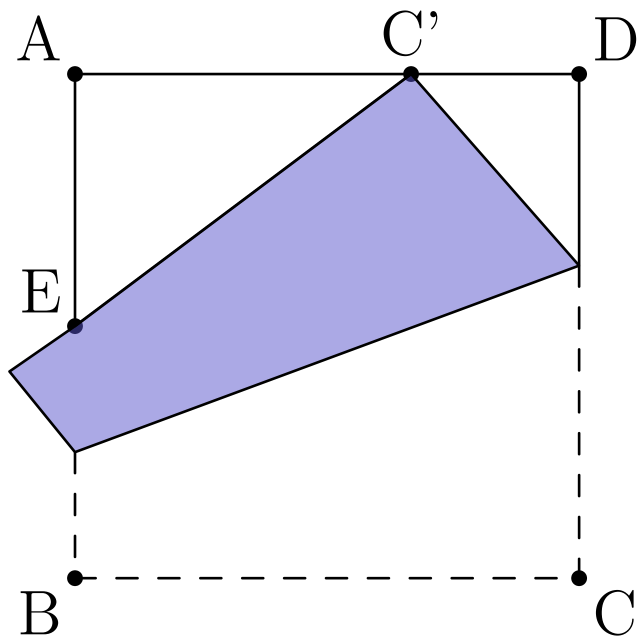
as it is an equilateral triangle. Since the height is $\sqrt{11}$ and the base is 2, the area is

$$\frac{2 \cdot \sqrt{11}}{2} = \sqrt{11}.$$

Thus, the total area is $\sqrt{12} + \sqrt{11}$. Therefore, $m + n = 23$.

Thus, the answer is **D**.

21. A square piece of paper has side length 1 and vertices A, B, C, and D in that order. As shown in the figure, the paper is folded so that vertex C meets edge $\overline{AD}$ at point C', and edge $\overline{BC}$ intersects edge $\overline{AB}$ at point E. Suppose that $C'D = \frac{1}{3}$. What is the perimeter of triangle $\triangle AEC'$?



A 2

B $1 + \dfrac{2}{3}\sqrt{3}$

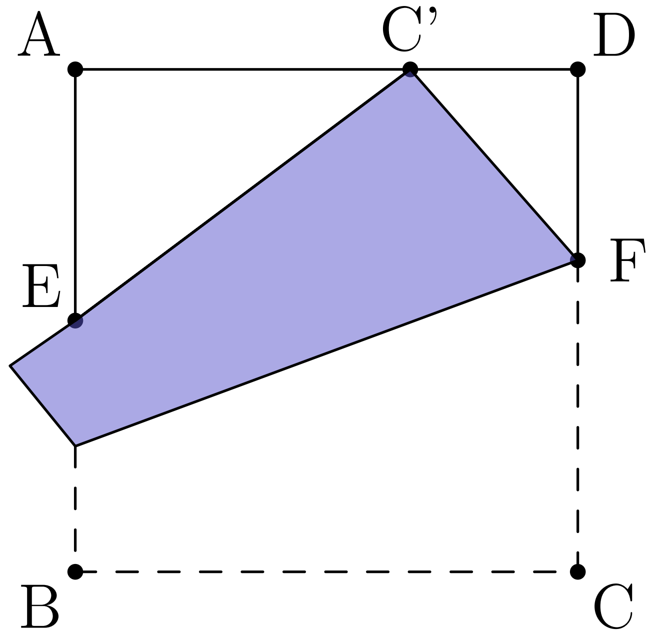
C $\dfrac{13}{6}$

D $1 + \dfrac{3}{4}\sqrt{3}$

E $\dfrac{7}{3}$

Solution(s):

Consider the following diagram, which is identical to the one in the problem, only with all the points labelled.



Since $\angle FC'E = 90^\circ$, we know

$$\angle FC'D = 90^\circ - EC'A.$$

Since $\angle C'AE = 90^\circ$, we know

$$\angle C'EA = 90^\circ - EC'A = \angle FC'D.$$

Therefore, by angle angle similarity, we know $AC'E \sim DFC'$. Therefore, the perimeter of AEC' is equal to the area of DFC' times $\frac{AC'}{DF}$.

Since $C'F + FD = 1$, the perimeter of DFC' is

$$\frac{1}{3} + 1 = \frac{4}{3}.$$

Also,

$$AC' = 1 - DC'$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}.$$

There, the perimeter of AEC' can be simplified to

$$\frac{4}{3} \cdot \frac{\frac{2}{3}}{DF} = \frac{8}{9 \cdot DF}.$$

By the Pythagorean theorem on DFC', we know

$$(C'D)^2 + (DF)^2 = (C'F)^2$$

$$= (1 - DF)^2.$$

This means

$$\frac{1}{3}^2 + DF^2 = 1 - 2DF + DF^2$$

so

$$1 - 2DF = \frac{1}{9}.$$

Therefore, $DF = \frac{4}{9}$.

This means the area is

$$\frac{8}{9 \cdot \frac{4}{9}} = \frac{8}{4}$$

$$= 2.$$

Thus, the answer is **A**.

22. Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. The probability that at least one box receives 3 blocks all of the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

A 47

B 94

C 227

D 471

E 542

Solution(s):

First, by WLOG, we can index the boxes by how Ang put his blocks in the boxes. Then, we can count the total number of ways to place all of Ben and Jasmin's blocks. This has a total of $(5!)^2$ ways it can be done.

Now, we find the total number of configurations that have at least one box that have the same color. We can do this with the principle of inclusion exclusion.

First, we get that there are $\binom{5}{1} \cdot (4!)^2$ distributions as there are $\binom{5}{1}$ ways to choose the box that has 3 of the same color, and there are $4!$ ways to choose the ordering for the other two boxes. However, this overcounts the number of configurations with 2 boxes that have 3 of the same block.

That would be counted as $\binom{5}{2} \cdot (3!)^2$ for similar reasons to above.

This also over counts the number of configurations with 3 boxes that are all the same color, so we add those back. We keep adding and removing configurations that are over and undercounted.

This leads to our number of configurations being

$$\frac{1}{(5!)^2}\left(\binom{5}{1} \cdot (4!)^2 + \binom{5}{2} \cdot (3!)^2\right.$$

$$\left.+\binom{5}{3} \cdot (2!)^2 + \binom{5}{4} \cdot (1!)^2\right.$$

$$+\binom{5}{5}\cdot(0!)^2\Bigg),$$

which is equal to

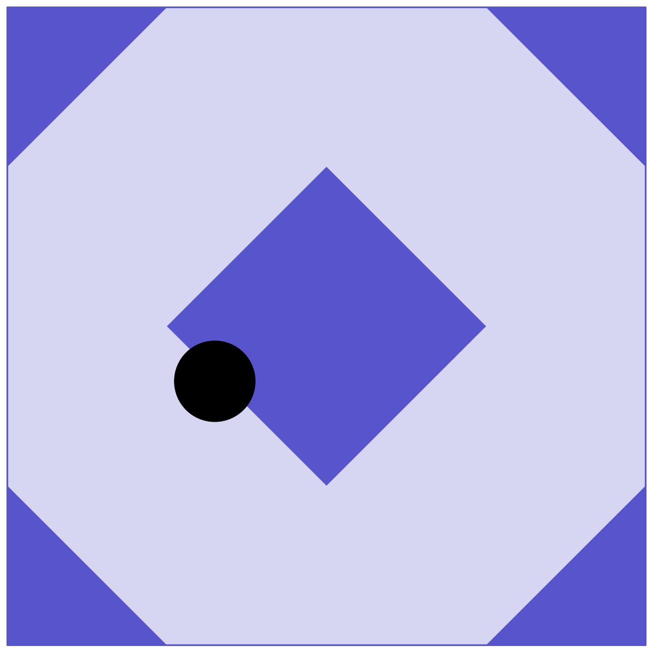
$$\frac{2256}{14400}=\frac{639}{3600}$$
$$=\frac{71}{400}.$$

This makes $m+n=471$.

Thus, the answer is **D**.

23. A square with side length 8 is colored a dark purple, except for 4 bright isosceles right triangular regions with legs of length 2 in each corner of the square and a bright diamond with side length $2\sqrt{2}$ in the center of the square, as shown in the diagram.

A circular coin with diameter 1 is dropped onto the square and lands in a random location where the coin is completely contained within the square. The probability that the coin will cover part of the darker region of the square can be written as $\frac{1}{196}\left(a+b\sqrt{2}+\pi\right)$, where a and b are positive integers. What is $a+b$?



A 64

B 66

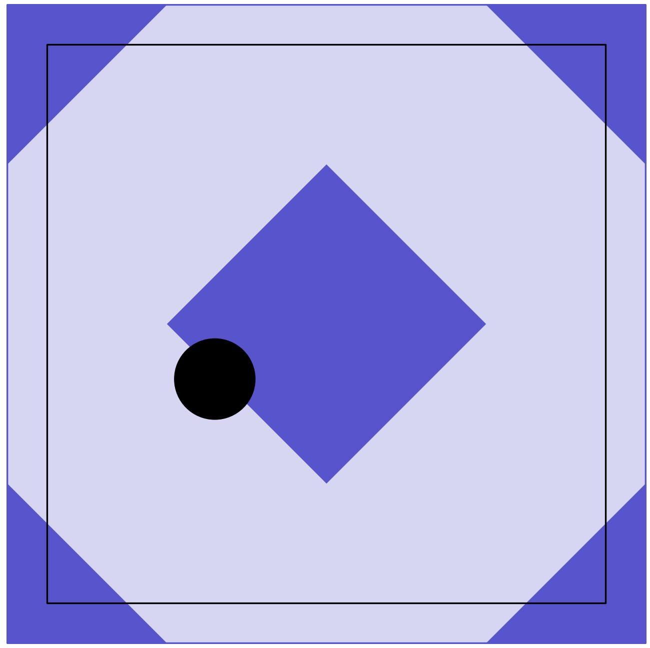
C 68

D 70

E 72

Solution(s):

The radius of the circle is $\frac{1}{2}$. This makes the total possible region for the circle's center to be in a square of length $8-\frac{1}{2}-\frac{1}{2}=7$. Shown below is the region in which the circle can be.



Now, the circle's center must be inside the black region or within $\frac{1}{2}$ of the black region.

For each of the corner, we can take a right triangle for this region. Now, we can find the altitude to the hypotenuse. This would be useful to find the as the area of a right triangle is equal to the altitude times the hypotenuse divided by 2, and the hypotenuse is twice the altitude, making the area equal to the altitude squared.

The altitude of this region is equal to the altitude of the portion of the right triangle in the black region and the blue region plus $\frac{1}{2}$. When finding the portion of the black right triangle in the blue region, we get a right triangle with length $2-\frac{1}{2}-\frac{1}{2}=1$. This has an altitude of $\frac{\sqrt{2}}{2}$. The total altitude therefore is $\frac{\sqrt{2}+1}{2}$. Thus, the answer is

$$(\frac{\sqrt{2}+1}{2})^2=\frac{3+2\sqrt{2}}{4}.$$

There are 4 corners, so the combined area of the right triangles is $3+2\sqrt{2}$.

For the center square, we find the area within it, the area that is within $\frac{1}{2}$ of the edge, and within $\frac{1}{2}$ of the corners. The area in the square is $(2\sqrt{2})^2=8$. The area within the $\frac{1}{2}$ of the edges is $\frac{1}{2}\cdot 2\sqrt{2}=\sqrt{2}$ since its a rectangle of with lengths $\frac{1}{2}$ and $2\sqrt{2}$. The total area from this is $4\sqrt{2}$. The area within $\frac{1}{2}$ of the points are 4 quarter circles of radius $\frac{1}{2}$, so their combined area is $\frac{\pi}{4}$. The total area is

$$3 + 2\sqrt{2} + 9 + 4\sqrt{2} + \frac{\pi}{4}$$

$$= 8 + 6\sqrt{2} + \frac{\pi}{4}.$$

The probability is

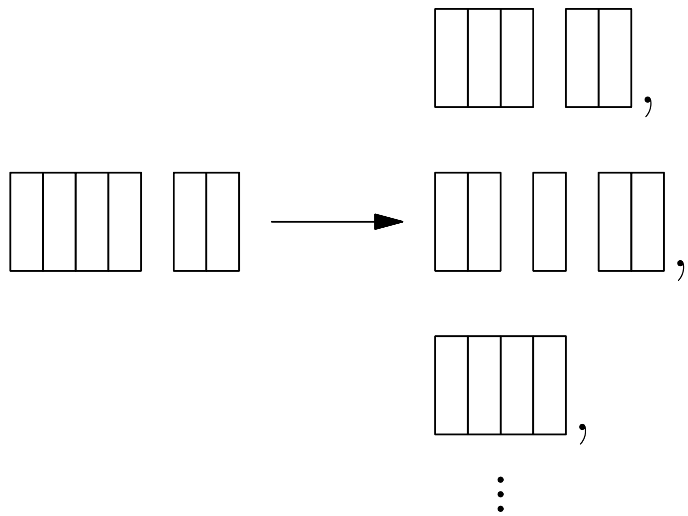
$$\frac{11 + 6\sqrt{2} + \frac{\pi}{4}}{49}$$

$$= \frac{44 + 24\sqrt{2} + \pi}{196}.$$

This makes the answer $44 + 24 = 68$.

Thus, the answer is **C**.

24. Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3,2),(2,1,2),(4),(4,1),(2,2)$, or $(1,1,2)$.



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

A $(6,1,1)$

B $(6,2,1)$

C $(6,2,2)$

D $(6,3,1)$

E $(6,3,2)$

Solution(s):

If Arjun can force the walls to have symmetry after his turn, in whichevery wall has matching wall of the same size, then he can always copy Beth's move on the matching wall. I claim that Arjun can do this with the setups from choices A,C,D, and E.

With A, Arjun can split it into $(2,2,1,1)$ which is symmetric.

With C, Arjun can split it into $(2,2,2,2)$ which is symmetric.

With D, Arjun can split it into $(3,2,3,2)$ which is symmetric.

With E, Arjun can split it into $(3,2,3,2)$ which is symmetric.

This leaves just B as a possible answer. Now we must prove it works. With $(6,2,1)$, Arjun can make

$$(6,2),(6,1),(6,1,1),(5,2,1),$$

$$(4,2,1),(4,2,1,1),(3,2,1,1),$$

$$(2,2,2,1),(3,1,2,1).$$

If given the last seven configurations by Arjun, Beth can make it symmetric by turning it into $(2,2,1,1)$, which is a winning situtation for her since it is symmetric. Thus, we must now verify that $(6,1)$ and $(6,2)$ are winning for her. Both of these lead to $(3,2,1)$, so it suffices to prove that $(3,2,1)$ is winning for Beth when given to Arjun.

With $(3,2,1)$, Arjun can make

$$(3,2),(3,1),(3,1,1),$$

$$(2,2,1),(1,2,1).$$

Each can be made symmetric by Beth in the following ways:

$(3,2) \implies (2,2)$

$(3,1) \implies (1,1)$

$(3,1,1) \implies (1,1,1,1)$

$(2,2,1) \implies (2,2)$

$(1,2,1) \implies (1,1)$

This makes Beth have a winning strategy with $(6,2,1)$.

Thus, the answer is **B**.

25. Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and 30, inclusive. Exactly 300 points in S lie on or below a line with equation $y = mx$. The possible values of m lie in an interval of length $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$?

A 31

B 47

C 62

D 72

E 85

Solution(s):

First, lets create a way to find the number of lattice points under $(31, y)$, if y is relatively prime with 31. With $(0, 0)$ and $(31, y)$, we can make a rectangle with $32(y + 1)$ points. There are $32(y + 1) - 2$ points not on the diagonal, and half of them are under the line. Also, there are y points under the line that aren't in S since their x-coordinate is 31 and 30 other points that aren't in S since their y-coordinate is 0. Therefore, if the line goes through $(31, y)$, then the total number of points is

$$\frac{32(y+1)-2}{2} - y - 30$$

$$= 15(y - 1).$$

If $y = 21$, then there are 300 points in S, making $m = \frac{21}{31}$ in the interval.

Also, notice that the number of lattice points is equal to

$$\sum_{i=1}^{30} \lfloor mx \rfloor.$$

As m increases, each term stays constant or increases. Since $m = \frac{21}{31}$ is in the interval, the lower bound can be found by finding the largest $m \leq \frac{21}{31}$ such that at least one term increases as m increases.

If a number was $\frac{a}{b}$ and $b \leq 30$, $\frac{a}{b} > 23$, then

$$\frac{a}{b} = \frac{2x+1}{3x+1}$$

if $b = 3x+1$,

$$\frac{a}{b} = \frac{2x+2}{3x+2}$$

if $b = 3x+2$, or

$$\frac{a}{b} = \frac{2x+3}{3x+3}$$

if $b = 3x+3$. These would all be greater than $\frac{21}{31}$.

Therefore, $\frac{2}{3}$ must be the lower bound. Also, we now know that the fractions from above are the possible upper bounds. Furthermore, a greater x would create a smaller interval, so we need to just look at the interval being

$$\frac{19}{28}, \frac{20}{29}, \frac{21}{30}.$$

The least of these is $\frac{19}{28}$, so the interval is of length

$$\frac{19}{28} - \frac{2}{3} = \frac{1}{84}.$$

This makes our answer 85.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc10/2021B