2021 AMC 10A Fall Solutions

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1. What is the value of

$$\frac{(2112-2021)^2}{169}?$$

A 7

B 21

C 49

D 64

E 91

Solution(s):

We can simplify the expression as follows:

$$\begin{aligned}\frac{(2112-2021)^2}{169} &= \frac{91^2}{169}\\ &= \frac{91^2}{13^2}\\ &= \left(\frac{91}{13}\right)^2\\ &= 7^2\\ &= 49.\end{aligned}$$

Thus, **C** is the correct answer.

2. Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

- A 16
- B 17
- C 18
- D 19
- E 20

Solution(s):

If she shortens the 4 unit side by 1, she has a 3×6 card, which has an area of $2 \cdot 6 = 18$.

If Menkara shortens the other side, she gets a $4 \cdot 5$ card, which has an area of $4 \cdot 5 = 20$.

In the problem, Menkara shortened the 4 unit side. Therefore, the area of the card if she shortened the other side is 20.

Thus, **E** is the correct answer.

3. What is the maximum number of balls of clay of radius 2 that can completely fit inside a cube of side length 6 assuming the balls can be reshaped but not compressed before they are packed in the cube?

- A 3
- B 4
- C 5
- D 6
- E 7

Solution(s):

The volume of the cube is $6^3 = 216$. The volume of a ball of clay is $\frac{4}{3}\pi 2^3 = \frac{32\pi}{3}$.

Since the balls can be reshaped but not compressed, the answer is

$$\left\lfloor \frac{216}{\frac{32\pi}{3}} \right\rfloor = \left\lfloor \frac{81\pi}{4} \right\rfloor.$$

Approximating using $\pi \approx 3.14$, we get

$$12 \leq 4\pi \leq 13$$

and

$$\left\lfloor \frac{81}{13} \right\rfloor \leq \left\lfloor \frac{81}{4\pi} \right\rfloor \leq \left\lfloor \frac{81}{12} \right\rfloor.$$

Since

$$\left\lfloor \frac{81}{12} \right\rfloor = \left\lfloor \frac{81}{12} \right\rfloor = 6,$$

we get that

$$\left\lfloor \frac{81}{4\pi} \right\rfloor = 6.$$

Thus, **D** is the correct answer.

4. Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

A $2\frac{3}{4}$

B $3\frac{3}{4}$

C $4\frac{1}{2}$

D $5\frac{1}{2}$

E $6\frac{3}{4}$

Solution(s):

Mr. Lopez would take

$$\frac{6}{30} \cdot 60 = 12$$

minutes to travel on Route A.

On Route B, he would take

$$\left(\frac{5 - .5}{40} + \frac{.5}{20}\right) \cdot 60 = 8.25$$

minutes.

The difference in times along these routes is $12 - 8.25 = 3.75$ minutes.

Thus, **B** is the correct answer.

5. The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A?

A 1

B 3

C 5

D 7

E 9

Solution(s):

Note that A cannot be even, as then the number would be divisible by 2.

A also cannot be 5, as that would make the number divisible by 5.

If A equaled 1 or 7, then the sum of the digits of the number would be 6 and 12 respectively.

This would make the number divisible by 3, so that rules out A equaling either of these numbers.

Finally, if A equals 3, then the whole number becomes 202109. If we look at the difference of the sums of alternating digits, we get

$$2 + 2 - 1 - 3 = 0,$$

which means the number is divisible by 11.

This means that A must be 9.

Thus, **E** is the correct answer.

6. Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

A 6

B 8

C 10

D 11

E 15

Solution(s):

There are 40 gaps between the 1 st and 41 st pole, which means that the distance between consecutive poles is

$$5280 \div 40 = 132$$

feet.

This means that each of Elmer's strides is

$$132 \div 44 = 3$$

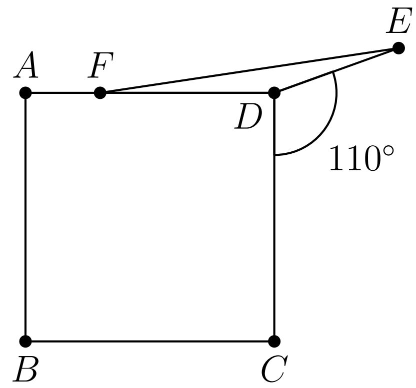
feet. Similarly, each of Oscar's strides is

$$132 \div 12 = 11$$

feet. This makes Oscar's leap $11 - 3 = 8$ feet longer.

Thus, **B** is the correct answer.

7. As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110°$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



A 160

B 164

C 166

D 170

E 174

Solution(s):

Since $\angle ADC = 90°$, we get that

$$\angle FDE = 360° - 90° - 110°$$
$$= 160°.$$

Also since $\triangle FDE$ is isosceles, we get that

$$\angle EFD = \frac{180° - 160°}{2} = 10°.$$

Finally, we get that

$$\angle AFE = 180° - 10° = 170°.$$

Thus, **D** is the correct answer.

8. A two-digit positive integer is said to be *cuddly* if it is equal to the sum of its nonzero tens digit and the square of its units digit. How many two-digit positive integers are cuddly?

A 0

B 1

C 2

D 3

E 4

Solution(s):

Let $\underline{a}\,\underline{b}$ be a 2-digit cuddly number.

Then

$$10a + b = a + b^2.$$

Rearranging, we get

$$9a = b(b-1).$$

This means that 9 divides either b or $b-1$ (3 cannot divide both b and $b-1$).

The only way this is possible is if $b = 9$ (b is one digit, so it can't be anything else). Checking, we get that 89 is a *cuddly* number. This shows that there is only 1 two-digit *cuddly* number.

Thus, **B** is the correct answer.

9. When a certain unfair die is rolled, an even number is 3 times as likely to appear as an odd number. The die is rolled twice. What is the probability that the sum of the numbers rolled is even?

A $\frac{3}{8}$

B $\frac{4}{9}$

C $\frac{5}{9}$

D $\frac{9}{16}$

E $\frac{5}{8}$

Solution(s):

Let p be the probability that an odd number is rolled. Then $3p$ is the probability an even number is rolled. We know that

$$p + 3p = 1 \Rightarrow p = \frac{1}{4}.$$

The only way for the sum to be even is if both rolls have the same parity. This happens with a probability of

$$\frac{1}{4}^2 + \frac{3}{4}^2 = \frac{10}{16} = \frac{5}{8}.$$

Thus, **E** is the correct answer.

10. A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are $50, 20, 20, 5,$ and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?

A -18.5

B -13.5

C 0

D 13.5

E 18.5

Solution(s):

Recall

$$\text{Expected value} =$$

$$\Sigma(\text{Outcome} \cdot \text{Probability}).$$

Therefore,

$$\begin{aligned} t &= \frac{1}{5}(50 + 20 + 20 + 5 + 5) \\ &= \frac{1}{5} \cdot 100 = 20 \end{aligned}$$

and

$$\begin{aligned} s = 50 \cdot \frac{50}{100} &+ 20 \cdot \frac{20}{100} \\ +20 \cdot \frac{20}{100} + 5 \cdot \frac{5}{100} &+ 5 \cdot \frac{5}{100} \\ = 25 + 4 + 4 + .25 &+ .25 = 33.5 \end{aligned}$$

$t - s = -13.5.$

Thus, **B** is the correct answer.

11. Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?

- A 70
- B 84
- C 98
- D 105
- E 126

Solution(s):

Let x be the length of the ship. Then in the time that Emily moves 210 steps, the ship moves $210 - x$ steps.

In the time that Emily moves 42 steps, the ship moves $x - 42$ steps. Since the ship and Emily move at a constant rate

$$\frac{210}{210 - x} = \frac{42}{x - 42}.$$

Cross-multiplying yields

$$210x - 210 \cdot 42 = 210 \cdot 42 - 42x$$

$$42 \cdot 6x = 2 \cdot 42 \cdot 210$$

$$x = 70.$$

Thus, **A** is the correct answer.

12. The base-nine representation of the number N is $27,006,000,052_{\text{nine}}$. What is the remainder when N is divided by 5?

A 0

B 1

C 2

D 3

E 4

Solution(s):

Note that

$$9 \equiv -1 \pmod 5.$$

Then if expand N using the definition of bases, we get

$$\begin{gathered} N = 2 \cdot 9^{10} + 7 \cdot 9^{9} + 6 \cdot 9^{6} + \\ 5 \cdot 9 + 2 \\ \equiv 2(-1)^{10} + 7(-1)^{9} + 6(-1)^{6} + \\ 5(-1) + 2 \pmod 5 \\ \equiv 2 - 7 + 6 - 5 + 2 \pmod 5 \\ \equiv 3 \pmod 5. \end{gathered}$$

Thus, **D** is the correct answer.

13. Each of 6 balls is randomly and independently painted either black or white with equal probability. What is the probability that every ball is different in color from more than half of the other 5 balls?

A $\frac{1}{64}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{5}{16}$

E $\frac{1}{2}$

Solution(s):

Note that for this restriction to hold, there must be 3 balls of each color.

There are $2^6 = 64$ ways to color the balls and $\binom{6}{3} = 20$ to choose which balls are white.

The desired probability is therefore $\frac{20}{64} = \frac{5}{16}$.

Thus, **D** is the correct answer.

14. How many ordered pairs (x, y) of real numbers satisfy the following system of equations?

$$x^2 + 3y = 9$$
$$(|x| + |y| - 4)^2 = 1$$

A 1

B 2

C 3

D 5

E 7

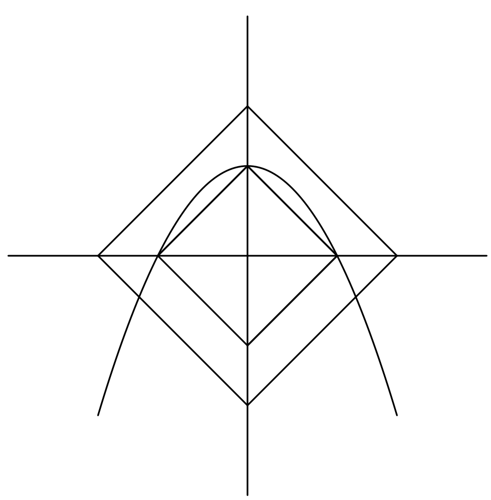
Solution(s):

The second equation seems very similar to that of a diamond. Rearranging, we get

$$|x| + |y| - 4 = \pm 1$$

$$|x| + |y| = \{3, 5\}.$$

We can graph this to see if we can figure out where the intersection points are.



From this, we can see that there are 5 intersection points.

Thus, **D** is the correct answer.

15. Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C?

A 24π

B 25π

C 26π

D 27π

E 28π

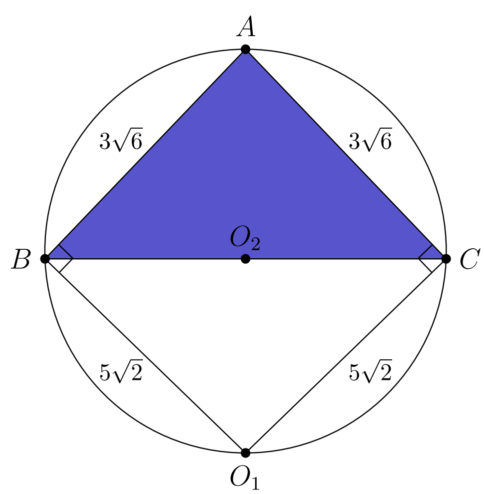
Solution(s):

Let $\odot C_1$ be the circle that is tangent to AB and AC. Then

$$\angle ABO_1 = \angle ACO_1 = 90^\circ,$$

making the two angles supplementary. This makes ABO_1C cyclic.

Let $\odot O_2$ be the circumcircle of ABO_1C. This makes $\odot O_2$ the circumcircle of $\triangle ABC$ as well.



We also know that AO_1 is the diameter of O_2 since AO_1 bisects $\angle BAC$.

By the Pythagorean theorem, we get that

$$\begin{aligned} AO_1 &= \sqrt{AB^2 + BO_1^2} \\ &= \sqrt{54 + 50} \\ &= 2\sqrt{26}. \end{aligned}$$

This makes the area of $\odot O_2$ 26π.

Thus, **C** is the correct answer.

16. The graph of

$$f(x) = |\lfloor x \rfloor| - |\lfloor 1 - x \rfloor|$$

is symmetric about which of the following? (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

A the y-axis

B the line $x = 1$

C the origin

D the point $\left(\frac{1}{2}, 0\right)$

E the point $(1, 0)$

Solution(s):

Note that

$$\begin{aligned} f(1-x) &= |\ \lfloor 1 - x \rfloor\ | - |\ \lfloor x \rfloor\ | \\ &= -f(x). \end{aligned}$$

This means that

$$f(\frac{1}{2} + x) = -f(\frac{1}{2} - x).$$

Therefore, the graph is symmetric about the point $(\frac{1}{2}, 0)$.

Thus, **D** is the correct answer.

17. An architect is building a structure that will place vertical pillars at the vertices of regular hexagon $ABCDEF$, which is lying horizontally on the ground. The six pillars will hold up a flat solar panel that will not be parallel to the ground. The heights of pillars at A, B, and C are 12, 9, and 10 meters, respectively. What is the height, in meters, of the pillar at E?

A 9

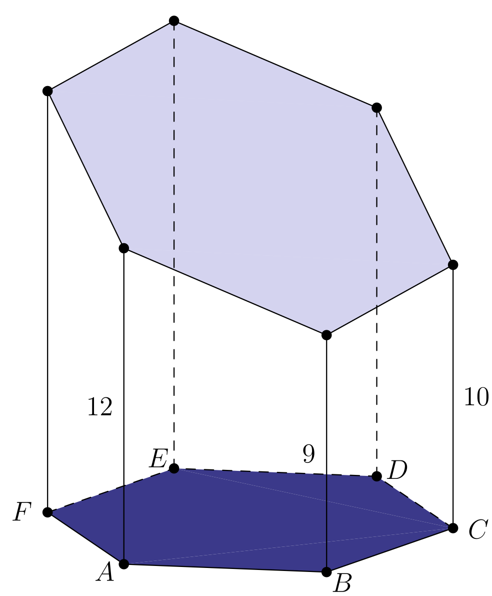
B $6\sqrt{3}$

C $8\sqrt{3}$

D 17

E $12\sqrt{3}$

Solution(s):



Note that the inclination from pillar A to pillar B is 3 since the solar panel is flat.

Let G be the center of the solar panel. Since $\overline{CG} \mid\mid \overline{BA}$, we get that the height at G is $10 + 3 = 13$.

We also know that the heights at B, G, and E are collinear. This makes the height of the pillar at E

$$9 + 4 + 4 = 17.$$

Thus, **D** is the correct answer.

18. A farmer's rectangular field is partitioned into 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?

A 12

B 64

C 84

D 90

E 144

Solution(s):

There are 2 cases.

Case 1 : the top-right and bottom-left sections have the same crop

There are 4 options for which crop is in those sections. The other two sections have 3 options for the crop since each crop has one restriction for crops next to it. This gives us

$$4 \cdot 3 \cdot 3 = 36$$

combinations.

Case 2 : the top-right and bottom-left sections have different crops

There are 4 options for which crop is in the top-left section. Then there are 3 options for the top-right and 2 for the bottom-left. This leaves 2 options for the bottom-right section. This gives us another

$$4 \cdot 3 \cdot 2 \cdot 2 = 48$$

configurations.

In total, there are $36 + 48 = 84$ combinations.

Thus, **C** is the correct answer.

19. A disk of radius 1 rolls all the way around the inside of a square of side length $s > 4$ and sweeps out a region of area A. A second disk of radius 1 rolls all the way around the outside of the same square and sweeps out a region of area $2A$. The value of s can be written as $a + \frac{b\pi}{c}$, where $a, b,$ and c are positive integers and b and c are relatively prime. What is $a + b + c$?

A 10

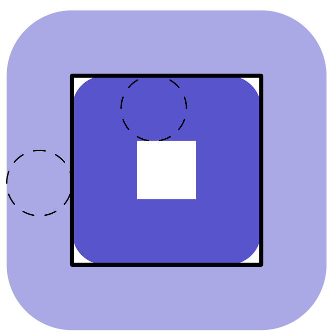
B 11

C 12

D 13

E 14

Solution(s):



The side length of the inner square traced out by the inner circle is $s - 4$.

There are also the small pieces remaining in the corner. These form a total area of

$$(1+1)^2 - \pi 1^2 = 4 - \pi.$$

Therefore,

$$A = s^2 - (s-4)^2 - (4-\pi)$$

$$= 8s - 20 + \pi.$$

The outer disk traces out an area that is comprised of 4 rectangles and 4 quarter-circles. The rectangles have area $s \cdot 2 = 2s$ and the quarter-circles form a circle with radius 2 and area 4π.

This gives us

$$2A = 8s + 4\pi.$$

Equating the two equations we get

$$8s + 4\pi = 2(8s - 20 + \pi).$$

Solving yields

$$8s = 40 + 2\pi$$

$$s = 5 + \frac{\pi}{4}.$$

Thus, **A** is the correct answer.

20. How many ordered pairs of positive integers (b, c) exist where both $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ do not have distinct, real solutions?

A 4

B 6

C 8

D 10

E 12

Solution(s):

Recall that the only way a quadratic does not have distinct, real solutions is if the discriminant is nonpositive.

This gives us that

$$b^2 \leq 4c \text{ and } c^2 \leq 4b.$$

Squaring the first inequality gives us that $b^4 \leq 16c^2$ since b and c are positive.

Multiplying the second inequality by 16 and combining gives us

$$b^4 \leq 16c^2 \leq 64b.$$

The only values of b that satisfy $b^4 \leq 64b$ are $1, 2, 3,$ and 4.

Case $1 : b = 1$

This gives us

$$1 \leq 16c^2 \leq 64.$$

Only $c = 1$ and $c = 2$ work.

Case $2 : b = 2$

This gives us

$$16 \leq 16c^2 \leq 128.$$

Only $c = 1$ and $c = 2$ work.

Case $3 : b = 3$

This gives us

$$81 \leq 16c^2 \leq 192.$$

Only $c = 3$ works.

Case $4 : b = 4$

This gives us

$$256 \leq 16c^2 \leq 256.$$

Only $c = 1$ works.

This gives us a total of 6 pairs that work.

Thus, **B** is the correct answer.

21. Each of 20 balls is tossed independently and at random into one of the 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

A 1

B 4

C 8

D 12

E 16

Solution(s):

For the sake of simplicity, we can assume the balls and bins are both distinguishable.

Since each case includes having 4 balls in 3 bins, we can leave those out during our calculation.

For p, there are 5 choices for the bin with 3 balls and then 4 choices for the bin with 5 balls. Finally, there are $\binom{8}{3} = 56$ ways to choose which balls go in the bins.

For q, after cancelling out 3 of the 4 s, there are $\binom{8}{4} = 70$ ways to ensure 4 balls go in each of the remaining bins.

Since the total number of distributions is the same for both p and q, we can let $\frac{p}{q}$ be the ratio of the numerators. Therefore,

$$\frac{p}{q} = \frac{20 \cdot 56}{70} = 16.$$

Thus, **E** is the correct answer.

22. Inside a right circular cone with base radius 5 and height 12 are three congruent spheres with radius r. Each sphere is tangent to the other two spheres and also tangent to the base and side of the cone. What is r?

A $\frac{3}{2}$

B $\frac{90-40\sqrt{3}}{11}$

C 2

D $\frac{144-25\sqrt{3}}{44}$

E $\frac{5}{2}$

Solution(s):

We can use coordinate geometry to solve this problem.

WLOG, let the origin be the center of the base of the cone. Then let the center of the one of the spheres be $(0, \frac{2r}{\sqrt{3}}, r)$. We get this y-coordinate since the centers of the spheres form an equilateral triangle.

We know that this sphere is internally tangent to the cone, so we know that it is tangent to the plane with equation

$$5x + 12y = 60.$$

The distance from the center of the sphere to this plane is then r.

Using the formula for the distance to a plane from a point, we get

$$r = \frac{\left|12 \cdot \frac{2r}{\sqrt{3}} + 5r - 60\right|}{\sqrt{0^2 + 5^2 + 12^2}}.$$

Solving for r, we get

$$r = \frac{\mid (5+8\sqrt{3})r - 60 \mid}{13}$$

$$13r = -(5+8\sqrt{3})r + 60$$

$$(8\sqrt{3}+18)r = 60$$

$$r = \frac{30}{4\sqrt{3}+9} \cdot \frac{9-4\sqrt{3}}{9-4\sqrt{3}}$$

$$r = \frac{30(9-4\sqrt{3})}{33}$$

$$r = \frac{90-40\sqrt{3}}{11}.$$

Thus, **B** is the correct answer.

23. For each positive integer n, let $f_1(n)$ be twice the number of positive integer divisors of n, and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$?

A 7

B 8

C 9

D 10

E 11

Solution(s):

First, let us see what values of x satisfy $f_1(x) = 12$. For this to happen, x must have 6 factors. This is only possible if $x = pq^2$ or $x = p^5$ where p and q are primes.

The only numbers less than 50 that work are

$$12, 18, 20, 28, 44, 45, \text{ and } 50.$$

Note that $f_1(12) = 12$. This means that if $f_i(x) = 12$, then $f_n(x) = 12$ when $n \geq i$.

This means that all the values listed above satisfy $f_{50}(x) = 12$. This also tells us that if some $f_i(x)$ equals any of the above numbers, then $f_{50}(x) = 12$.

The only possibilities such that $x \leq 50$ is if $f_1(x)$ is either 18 or 20. This means x must have either 9 or 10 factors.

This means x is of the form p^2q^2 or p^4q. The only numbers less than 50 that work are 36 and 48.

We have exhausted all the possible values n such that $f_{50}(n) = 12$. This gives us 10 total solutions.

Thus, **D** is the correct answer.

24. Each of the 12 edges of a cube is labeled 0 or 1. Two labelings are considered different even if one can be obtained from the other by a sequence of one or more rotations and/or reflections. For how many such labelings is the sum of the labels on the edges of each of the 6 faces of the cube equal to 2?

A 8

B 10

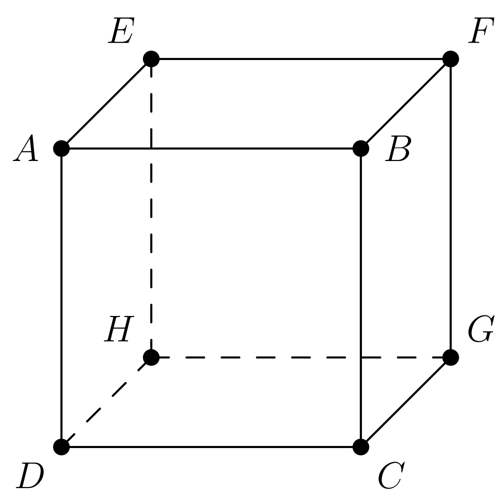
C 12

D 16

E 20

Solution(s):

Label the cube as follows.



Note that each face must have 2 zeros and 2 ones.This means that for all 6 faces, there are 6 zeros and 6 ones.

We can case on the sides of $ABCD$.

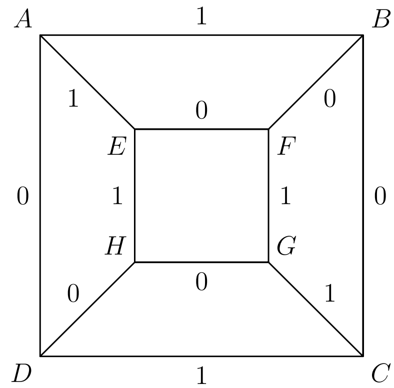
Case 1 : opposite sides have the same label

This gives us 2 ways to label the edges of $ABCD$. WLOG, let $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$ be labeled $1, 0, 1, 0$ respectively. We multiply by 2 at the end to take care of the other case.

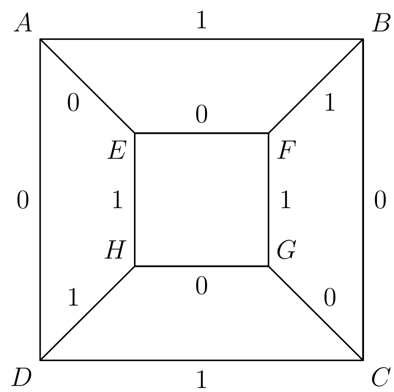
Then we can apply casework to the label of $\overline{AE}$.

If its label is 1, then we know that the label of $\overline{EF}$ and $\overline{BF}$ is 0 to satisfy the condition for the top face.

Then $\overline{FG}$ and $\overline{CG}$ must be 1. This forces $\overline{DH}$ and $\overline{GH}$ to be 0. Finally, $\overline{EH}$ must be 0.



If $\overline{AE}$ is 0, we can walk through all the faces as before, which will tell us that there is only 1 possible case in this scenario.



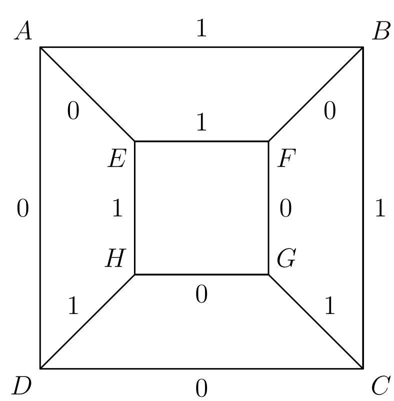
Therefore, this cases has $2 \cdot 2 = 4$ possible labelings.

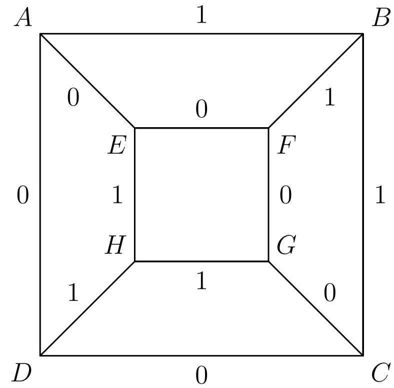
Case 2 : opposite edges have different labels

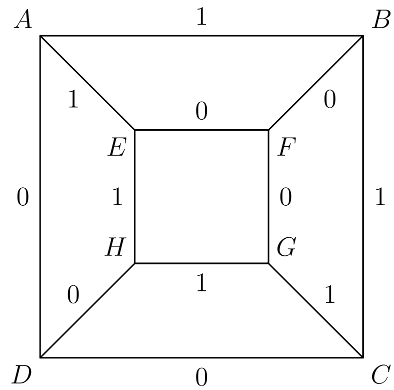
There are 4 ways to label the faces of $ABCD$. WLOG, label $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$ be labeled $1, 1, 0, 0$ respectively. We can multiply by 4 at the end for the other cases.

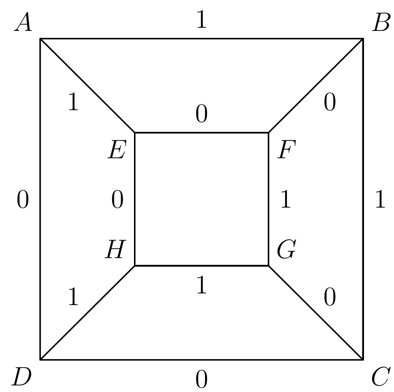
Now we case on the labels of $\overline{AE}$ and $\overline{BF}$.

As above, we can go through each pair of labels to see that each pair only gives us one possible labeling of the cube. There are 4 pairs, so this gives us 4 configurations.









Therefore, this case has $4 \cdot 4 = 16$ possible labelings.

Finally, there are a total of $4 + 16 = 20$ labelings.

Thus, **E** is the correct answer.

25. A quadratic polynomial with real coefficients and leading coefficient 1 is called *disrespectful* if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

A $\frac{5}{16}$

B $\frac{1}{2}$

C $\frac{5}{8}$

D 1

E $\frac{9}{8}$

Solution(s):

Let r and s be the roots of $\tilde{p}(x)$. Then

$$\tilde{p}(x) = (x-r)(x-s)$$
$$= x^2 - (r+s) + rs.$$

Note that the solutions to $\tilde{p}(\tilde{p}(x)) = 0$ are the solutions to

$$\tilde{p}(x) - r =$$
$$x^2 - (r+s)x + (rs - r) = 0$$

and

$$\tilde{p}(x) - s =$$
$$x^2 - (r+s)x + (rs - s) = 0.$$

Since there are only 3 distinct solutions, one of these quadratics must have a double root, and the other has to have 2 distinct roots.

WLOG, let the first equation be the one with a double root. Then we know that its discriminant is 0. This give us

$$(r+s)^2 = 4rs - 4r$$
$$r - s = \pm 2\sqrt{-r}.$$

For the other equation to have 2 solutions, its discriminant must be positive.

$$(r+s)^2 - 4rs + 4s > 0$$
$$(r-s)^2 > -4s$$
$$-4r > -4s$$
$$r - s < 0$$

From above, we can conclude that $r - s = -2\sqrt{r}$.

We know that the sum of the roots of $\tilde{p}(x)$ is

$$r + s = 2r + 2\sqrt{-r}.$$

This is maximized when $r = -\frac{1}{4}$, yielding $s = \frac{3}{4}$. Then

$$\tilde{p}(1) = 1^2 - \frac{1}{2} \cdot 1 - \frac{3}{16} = \frac{5}{16}.$$

Thus, **A** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2021C