2022 AMC 10A Solutions

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1. What is the value of

$$3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3}}}?$$

A $\frac{31}{10}$

B $\frac{49}{15}$

C $\frac{33}{10}$

D $\frac{109}{33}$

E $\frac{15}{4}$

Solution(s):

We can simplify this expression as follows:

$$\begin{aligned}3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3}}} &= 3+\cfrac{1}{3+\cfrac{1}{\cfrac{10}{3}}}\\ &= 3+\cfrac{1}{3+\cfrac{3}{10}}\\ &= 3+\cfrac{1}{\cfrac{33}{10}}\\ &= 3+\frac{10}{33}\\ &= \frac{109}{33}.\end{aligned}$$

Thus, **D** is the correct answer.

2. Mike cycled 15 laps in 57 minutes. Assume he cycled at a constant speed throughout. Approximately how many laps did he complete in the first 27 minutes?

- A 5
- B 7
- C 9
- D 11
- E 13

Solution(s):

We can set up a proportion to solve this problem:

$$\frac{15}{57} = \frac{x}{27}.$$

Cross multiplying, we get

$$x = \frac{15}{57} \cdot 27 = \frac{135}{19} \approx 7.$$

Thus, **B** is the correct answer.

3. The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

A 1

B 2

C 3

D 4

E 5

Solution(s):

Let x, y, and z be the three numbers. The conditions from the problem give us the following relations:

\begin{gather*} x + y + z = 96 \tag*{(1)} \\ x = 6z \tag*{(2)} \\ z = y - 40 \tag*{(3)}. \end{gather*}

Rearranging (3), we get $y = z + 40$. Plugging this new equation and (2) into (1), we get

$$6z + z + 40 + z = 96$$

$$8z + 40 = 96$$

$$8z = 56 \Rightarrow z = 7.$$

From this, we get that

$$x = 6 \cdot z = 6 \cdot 7 = 42$$

and

$$y = z + 40 = 7 + 40 = 47.$$

Therefore, $y - x = 47 - 42 = 5$.

Thus, **E** is the correct answer.

4. In some countries, automobile fuel efficiency is measured in liters per 100 kilometers while other countries use miles per gallon. Suppose that 1 kilometer equals m miles, and 1 gallon equals l liters. Which of the following gives the fuel efficiency in liters per 100 kilometers for a car that gets x miles per gallon?

A $\dfrac{x}{100lm}$

B $\dfrac{xlm}{100}$

C $\dfrac{lm}{100x}$

D $\dfrac{100}{xlm}$

E $\dfrac{100lm}{x}$

Solution(s):

We can do the following conversions to get the desired answer.

$$\frac{x \text{ mi}}{1 \text{ gal}} \cdot \frac{1 \text{ km}}{m \text{ mi}} = \frac{x \text{ km}}{m \text{ gal}}$$
$$\frac{x \text{ km}}{m \text{ gal}} \cdot \frac{1 \text{ gal}}{l \text{ L}} = \frac{x \text{ km}}{ml \text{ L}}$$
$$\frac{x \text{ km}}{ml \text{ L}} \cdot \frac{100 \text{ km}}{100 \text{ km}} = \frac{100x \text{ km}}{100ml \text{ L}}$$

Taking the reciprocal, we find that the equivalent fuel efficiency would be

$$\frac{100ml \text{ L}}{100x \text{ km}} = \frac{\frac{100ml}{x} \text{ L}}{100 \text{ km}}.$$

Thus, **E** is the correct answer.

5. Square $ABCD$ has side length 1. Points P, Q, R, and S each lie on a side of $ABCD$ such that $APQCRS$ is an equilateral convex hexagon with side length s. What is s?

A $\frac{\sqrt{2}}{3}$

B $\frac{1}{2}$

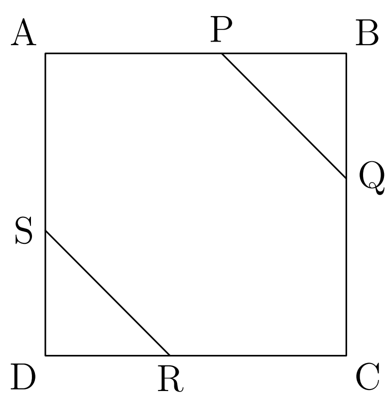
C $2-\sqrt{2}$

D $1-\frac{\sqrt{2}}{4}$

E $\frac{2}{3}$

Solution(s):

Consider the diagram:



Since $AP = QC = s$, we know that $PB = PQ$. This shows that $\triangle PBQ$ is a right triangle. Using the Pythagorean theorem, we get that $PB = \frac{s}{\sqrt{2}}$.

We also know that

$$1 = AB = AP + PB = s + \frac{s}{\sqrt{2}}.$$

This equation simplifies to

$$1 = (1 + \frac{1}{\sqrt{2}})s$$

Which implies that

$$s = \frac{1}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}+1}.$$

We can rationalize this fraction to get

$$\frac{\sqrt{2}}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = 2-\sqrt{2}.$$

Thus, **C** is the correct answer.

6. Which expression is equal to

$$\left|a-2-\sqrt{(a-1)^2}\right|$$

for $a < 0$?

A $3-2a$

B $1-a$

C 1

D $a+1$

E 3

Solution(s):

By the definition of square root, we get that

$$\sqrt{(a-1)^2} = |a-1|.$$

Since $a < 0$, we get that $a-1 < 0$, which means that $|a-1| = 1-a$.

The whole expression therefore simplifies to

$$|a-2-(1-a)| = |2a-3|.$$

Since $a < 0$, we know that $2a-3 < 0$. This means that $|2a-3| = 3-2a$.

Thus, **A** is the correct answer.

7. The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

A 3

B 6

C 8

D 9

E 12

Solution(s):

We can see that $180 = 2^2 \cdot 3^2 \cdot 5$ and $18 = 2 \cdot 3^2$.

This means that n must include a factor of 2^2 and 5. We also know that $45 = 3^2 \cdot 5$ and $15 = 3 \cdot 5$, which means that n has a factor of 3^2 and 5. Therefore, we can set

$$n = 2^2 \cdot 3 \cdot 5 = 60.$$

Thus, **B** is the correct answer.

8. A data set consists of 6 (not distinct) positive integers: $1, 7, 5, 2, 5,$ and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X?

A 10

B 26

C 32

D 36

E 40

Solution(s):

The average of the 6 numbers is

$$\frac{1+7+\cdots+X}{6}=\frac{20+X}{6}.$$

This value can equal any of the terms in the set, so we can case on what it equals.

$$\frac{20+X}{6}=1 \iff X=-14$$

$$\frac{20+X}{6}=7 \iff X=22$$

$$\frac{20+X}{6}=5 \iff X=10$$

$$\frac{20+X}{6}=2 \iff X=-8$$

$$\frac{20+X}{6}=X \iff X=4$$

Adding up all the positive values for X, we get 36.

Thus, **D** is the correct answer.

9. A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?

A 120

B 270

C 360

D 540

E 720

Solution(s):

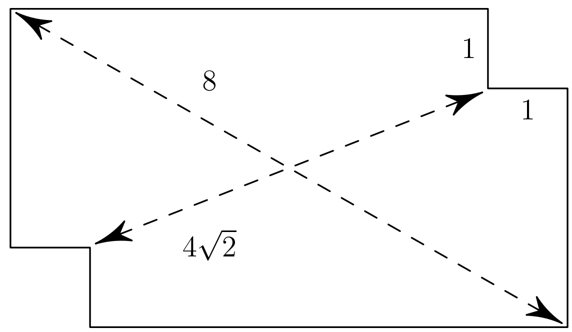
There are 5 choices for the color of the bottom left rectangle. This forces there to be 4 choices for the top left rectangle. The middle bottom rectangle touches both of the previous ones, so there are 3 color options for this rectangle.

The rectangle in the top right is also limited to 3 colors since it touches the two previous rectangles. Finally, the rectangle in the bottom right also has 3 color options.

Multiplying these together, we get 540 total colorings.

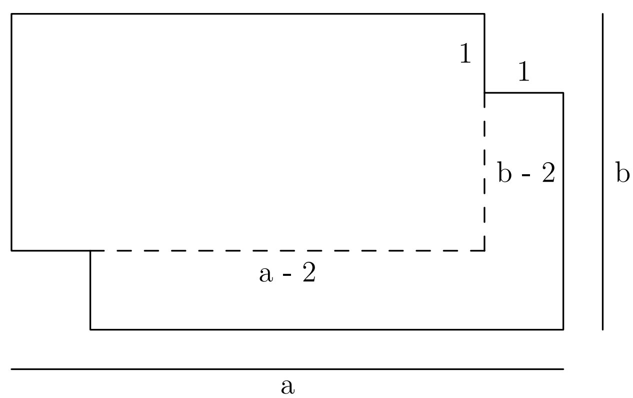
Thus, **D** is the correct answer.

10. Daniel finds a rectangular index card and measures its diagonal to be 8 centimeters. Daniel then cuts out equal squares of side 1 cm at two opposite corners of the index card and measures the distance between the two closest vertices of these squares to be $4\sqrt{2}$ centimeters, as shown below. What is the area of the original index card?



- A 14
- B $10\sqrt{2}$
- C 16
- D $12\sqrt{2}$
- E 18

Solution(s):



We can label a and b as the width and height as in the diagram. Then we get that

$$a^2 + b^2 = 64$$

and

$$(a-2)^2 + (b-2)^2 = 32.$$

The latter expression simplifies to

$$a^2 + b^2 - 4a - 4b + 4 + 4 = 32,$$

which is the same as

$$72 - 4(a + b) = 32.$$

From this we get

$$a + b = 10.$$

Squaring this, we get

$$a^2 + b^2 + 2ab = 100,$$

which gets us that

$$2ab = 36,$$

which means that the area (ab) is 18.

Thus, **E** is the correct answer.

11. Ted mistakenly wrote

$$2^m \cdot \sqrt{\frac{1}{4096}}$$

as

$$2 \cdot \sqrt[m]{\frac{1}{4096}}.$$

What is the sum of all real numbers m for which these two expressions have the same value?

A 5

B 6

C 7

D 8

E 9

Solution(s):

We can rewrite 4096 as 2^{12}, so $\dfrac{1}{4096} = 2^{-12}$. Then if we equate the given expressions, we get

$$2^m \cdot 2^{-6} = 2 \cdot 2^{\frac{-12}{m}}.$$

Equating the exponents, we get

$$m - 6 = 1 + \frac{-12}{m}.$$

Multiplying by m, we get

$$m^2 - 6m = m - 12$$

and so

$$m^2 - 7m + 12 = 0$$

$$(m-4)(m-3)$$

$$m=4,\ m=3$$

Therefore, we can see that the sum of the solutions is 7.

Thus, **C** is the correct answer.

12. On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

A 7

B 12

C 21

D 27

E 31

Solution(s):

For the first question, the truth-tellers will respond yes, the liars will respond yes, and the alternaters who decided to lie first will say yes. The alternaters who decide to tell the truth first will say no. Denote this as

$$22 = t + l + a_l.$$

For the second questions, the liars will respond yes, and the alternaters who decided to lie first will say yes (they are forced to tell the truth for this question). The truth-tellers will respond no, and the alternaters who told the truth first would lie this round, responding no. Denote this as

$$15 = l + a_l.$$

From this, we get that $t = 7$. The principal only gives candy to children who always tell the truth in the first round, therefore only giving them 7 candies total.

Thus, **A** is the correct answer.

13. Let $\triangle ABC$ be a scalene triangle. Point P lies on $\overline{BC}$ so that $\overline{AP}$ bisects $\angle BAC$. The line through B perpendicular to $\overline{AP}$ intersects the line through A parallel to $\overline{BC}$ at point D. Suppose $BP = 2$ and $PC = 3$. What is AD?

A 8

B 9

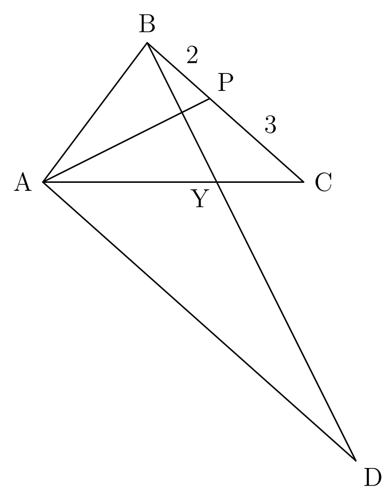
C 10

D 11

E 12

Solution(s):

Consider the following diagram:



By the Angle Bisector Theorem, we can label AB as $2x$ and AY as $3x$.

We also get that $\triangle ABY$ is isosceles since $AP \perp BY$. Therefore, $AY = AB = 2x$.

Since AD and BC are parallel, we know that

$$\triangle BYC \sim \triangle DYA.$$

$$YC = AC - AY = 3x - 2x = x,$$

so $AY = 2YC$.

Using similar triangles, we get that $AD = 2BC = 10$.

Thus, **C** is the correct answer.

14. How many ways are there to split the integers 1 through 14 into 7 pairs such that in each pair, the greater number is at least 2 times the lesser number?

A 108

B 120

C 126

D 132

E 144

Solution(s):

We can see that the numbers $1-8$ must be in different pairs. 7 must also be paired with 14 since no other number is at least twice 7.

Now let's look at what the other numbers can pair with. 8 and 9 can pair with any number $1-4$. 10 and 11 can pair with any number $1-5$, and 12 and 13 can pair with any number $1-6$.

8 can pair with 4 numbers, but then 9 only has 3 options since 8 took one. 10 then has 3 options, since 2 choices are taken, but it has one more to choose from (5). 11 then has 2 options, 12 has 2 options, and 13 only has 1.

Multiplying these together yields

$$4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 144.$$

Thus, **E** is the correct answer.

15. Quadrilateral $ABCD$ with side lengths $AB = 7, BC = 24, CD = 20$, $DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where a, b, and c are positive integers such that a and c have no common prime factor. What is $a + b + c$?

A 260

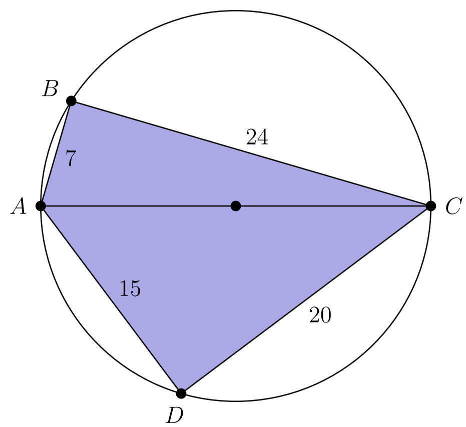
B 855

C 1235

D 1565

E 1997

Solution(s):



Notice that $7^2 + 24^2$ and $15^2 + 20^2$ are both the same. This forces $AC = 25$ since otherwise $\angle B$ and $\angle D$ would both be acute or obtuse, violating the fact that their sum is 180°.

Also since $\angle B$ is right, we know that AC is the diameter of the circle. The area of the circle is then $\frac{625}{4}\pi$.

To find the area of the quadrilateral, we can find the area of each of the triangles, which is

$$\frac{1}{2}(7 \cdot 24 + 20 \cdot 15) =$$

$$84 + 150 = 234.$$

To find the area outside the quadrilateral, we subtract to get

$$\frac{625}{4}\pi - 234 = \frac{625\pi - 936}{4}.$$

Therefore,

$$a + b + c = 625 + 936 + 4$$

$$= 1565.$$

Thus, **D** is the correct answer.

16. The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

A $\frac{24}{5}$

B $\frac{42}{5}$

C $\frac{81}{5}$

D 30

E 48

Solution(s):

Let $h, l,$ and w be the dimensions of the old box. Then the volume of the new box is

$$(h+2)(l+2)(w+2).$$

Expanding, we get

$$hlw + 2(hl + hw + lw)$$
$$+4(l + h + w) + 8.$$

We can use Vieta's formulas to find the terms in this expression. We get that

$$hlw = -\frac{D}{A} = \frac{3}{5},$$
$$hl + hw + lw = \frac{C}{A} = \frac{29}{10},$$

and

$$l + h + w = -\frac{B}{A} = \frac{39}{10}.$$

Plugging these values into the expression, we get

$$\frac{3}{5}+2\cdot\frac{29}{10}+4\cdot\frac{39}{10}+8=30.$$

Thus, **D** is the correct answer.

17. How many three-digit positive integers $\underline{a}\,\underline{b}\,\underline{c}$ are there whose nonzero digits $a, b,$ and c satisfy

$$0.\overline{\underline{a}\,\underline{b}\,\underline{c}} = \frac{1}{3}(0.\overline{a} + 0.\overline{b} + 0.\overline{c})?$$

(The bar indicates repetition, thus $0.\overline{\underline{a}\,\underline{b}\,\underline{c}}$ in the infinite repeating decimal $0.\underline{a}\,\underline{b}\,\underline{c}\,\underline{a}\,\underline{b}\,\underline{c}\,\cdots$)

- A 9
- B 10
- C 11
- D 13
- E 14

Solution(s):

Let's find a closed form expression for each of the repeating decimals. We can write $0.\overline{\underline{a}\,\underline{b}\,\underline{c}}$ as

$$0.\underline{a}\,\underline{b}\,\underline{c} + 0.000\,\underline{a}\,\underline{b}\,\underline{c} + \cdots.$$

From this, we can see that this an infinite geometric sequence with first term $0.\underline{a}\,\underline{b}\,\underline{c}$ and ratio $\dfrac{1}{1000}$.

Using the formula for the sum of a infinite geometric sequence, we get that this equals

$$\frac{0.\underline{a}\,\underline{b}\,\underline{c}}{1 - \dfrac{1}{1000}} = \frac{\underline{a}\,\underline{b}\,\underline{c}}{999}.$$

Similarly, note that we can write $0.\overline{a}$ as

$$0.a + 0.0a + 0.00a + \cdots.$$

As above, this equals

$$\frac{0.a}{1-\frac{1}{10}} = \frac{a}{9}.$$

Therefore,

$$0.\bar{b} = \frac{b}{9} \text{ and } 0.\bar{c} = \frac{c}{9}.$$

Substituting all these values into the condition, we get

$$\frac{\underline{a}\,\underline{b}\,\underline{c}}{999} = \frac{1}{3} \cdot \frac{a+b+c}{9}.$$

Multiplying through by 999 yields

$$\underline{a}\,\underline{b}\,\underline{c} = 37(a+b+c).$$

Note that we can express $\underline{a}\,\underline{b}\,\underline{c}$ as $100a + 10b + c$. Substituting this in, we get

$$100a + 10b + c = 37(a+b+c),$$

which simplifies to

$$63a = 27b + 36c \Rightarrow 7a = 3b + 4c.$$

All the solutions where $a = b = c$ work. The expression $3b + 4c$ remains constant if we increase b by 4 and decrease c by 4. We could also decrease b by 4 and increase c by 3.

Applying this principles to the first 9 triples yields

$$(4,8,1),(5,1,8),(5,9,2),(6,2,9)$$

as 4 more solutions. Therefore, there are a total of 13 solutions.

Thus, **D** is the correct solution.

18. Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \cdots, T_n$ returns the point $(1, 0)$ back to itself?

A 359

B 360

C 719

D 720

E 721

Solution(s):

Since we are working with angles and reflections, working with polar coordinates would make this problem easier to deal with.

Let (r, θ) be a polar coordinate. Rotating this by k degrees counterclockwise maps the point to $(r, \theta + k^\circ)$ and then reflecting it maps it to $(r, 180 - \theta - k^\circ.)$

Therefore, we have that

$$T_k(r, \theta) = (r, 180 - \theta - k^\circ).$$

From this, we can see that

$$T_{k+1}(T_k(r, \theta)) =$$

$$T_{k+1}(r, 180^\circ - \theta - k^\circ) =$$

$$(r, \theta - 1^\circ).$$

Now, let's analyze what happens to the point $(1, 0^\circ)$.

After T_1, we get $(1, 179^\circ)$.

After T_2, we get $(1, -1^\circ)$.

After T_3, we get $(1, 178^\circ)$.

After T_4, we get $(1, -2^\circ)$.

$$\vdots$$

After T_{2n-1}, we get $(1, 180^\circ - k^\circ)$.

After T_{2n}, we get $(1, -k^\circ)$.

From this, we can see that the first time the angle is back to 0° is when $n = 180$ and $k = 359$.

Thus, **A** is the correct answer.

19. Define L_n as the least common multiple of all the integers from 1 to n inclusive. There is a unique integer h such that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{17}=\frac{h}{L_{17}}$$

What is the remainder when h is divided by 17?

A 1

B 3

C 5

D 7

E 9

Solution(s):

We can combine all the addends on the left side into one fraction by making all their denominators 17!.

$$\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{17}=$$
$$\frac{\frac{17!}{1}+\frac{17!}{2}+\cdots+\frac{17!}{17}}{17!}$$

Therefore,

$$h=L_{17}\frac{\frac{17!}{1}+\frac{17!}{2}+\cdots+\frac{17!}{17}}{17!}.$$

We want to find h mod 17, so we can analyze the expression for h mod 17.

Each of

$$\frac{\frac{17!}{1}}{17},\frac{\frac{17!}{2}}{17!},\ldots,\frac{\frac{17!}{16}}{17!}$$

do not have a factor of 17 in the denominator. This means that when we multiply each of them by L_{17}, the resulting product will have a factor of 17 (17 divides L_{17}). That means all of these terms evaluate to 0 mod 17.

Now we just need to evaluate $\frac{L_{17}}{17}$ mod 17.

To calculate L_{17}, note that it must contain the highest power of every prime less than or equal to 17. Therefore,

$$L_{17} = 16 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17.$$

Now we simplify:

$$\begin{aligned}
&16 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \pmod{17} \\
&\equiv -1 \cdot 9 \cdot 35 \cdot 11 \cdot 13 \pmod{17} \\
&\equiv 9 \cdot 11 \cdot -13 \pmod{17} \\
&\equiv 99 \cdot 4 \pmod{17} \\
&\equiv -3 \cdot 4 \pmod{17} \\
&\equiv 5 \pmod{17}
\end{aligned}$$

Thus, **C** is the correct answer.

20. A four-term sequence is formed by adding each term of a four-term arithmetic sequence of positive integers to the corresponding term of a four-term geometric sequence of positive integers. The first three terms of the resulting four-term sequence are 57, 60, and 91. What is the fourth term of this sequence?

A 190

B 194

C 198

D 202

E 206

Solution(s):

Let the arithmetic sequence be

$$a, a+d, a+2d, a+3d$$

and the geometric sequence be

$$b, br, br^2, br^3.$$

Then

$$a+b=57, \tag{1}$$

$$a+d+br=60, \tag{2}$$

and

$$a+2d+br^2=91. \tag{3}$$

Subtracting (1) from (2) and (2) from (3), we get

$$d+b(r-1)=3$$

and

$$d+br(r-1)=31.$$

Subtracting these, we get

$$b(r-1)^2 = 28.$$

Since every variable is an integer, we get that $b = 28$ or $b = 7$.

If $b = 28$, then $r = 2$, $a = 29$, and $d = 25$. This forces the arithmetic sequence to be

$$29, 4, -21, -46,$$

which is a contradiction.

Therefore, $b = 7$. Then $r = 3$, $a = 50$, and $d = -11$. The arithmetic sequence is

$$50, 39, 28, 17,$$

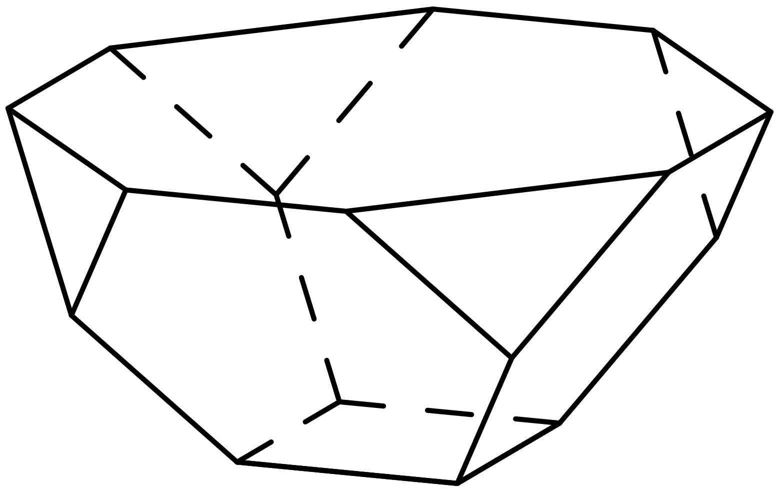
and the geometric sequence is

$$7, 21, 63, 189.$$

The desired answer is $17 + 189 = 206$.

Thus, **E** is the correct answer.

21. A bowl is formed by attaching four regular hexagons of side 1 to a square of side 1. The edges of the adjacent hexagons coincide, as shown in the figure. What is the area of the octagon obtained by joining the top eight vertices of the four hexagons, situated on the rim of the bowl?



A 6

B 7

C $5+2\sqrt{2}$

D 8

E 9

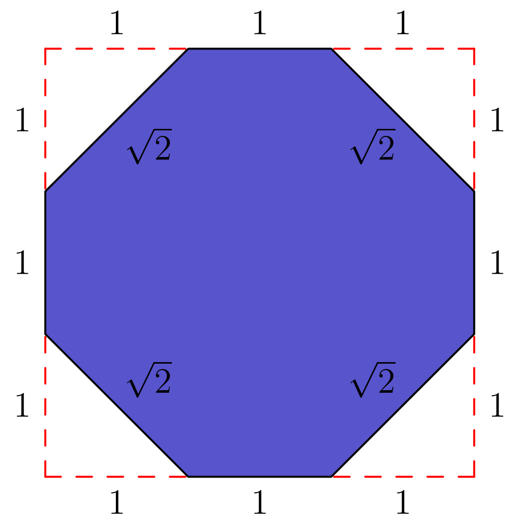
Solution(s):

We can extend line segments $l, m,$ and n as follows.

They are concurrent since l and m intersect and l and n intersect (they are on the same plane). Since l only intersects the plane of m and n once, it must intersect them both at that one point.

The dashed red lines create equilateral triangles on the lateral faces of the bowl, which all have side length 1. In the top plane, we know that $m \perp n$, so the dashed red lines create an isosceles right triangle with leg length 1.

The octagon looks like the diagram below.



The area of the octagon is the area of the square minus each of the four corner triangles. This is equal to

$$3^2 - 4(\frac{1}{2} \cdot 1^2) = 7.$$

Thus, **B** is the correct answer.

22. Suppose that 13 cards numbered $1, 2, 3, \cdots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards $1, 2, 3$ are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, $7, 8, 9, 10$ on the fourth pass, and $11, 12, 13$ on the fifth pass. For how many of the $13!$ possible orderings of the cards will the 13 cards be picked up in exactly two passes?

$$\boxed{7}\ \boxed{11}\ \boxed{8}\ \boxed{6}\ \boxed{4}\ \boxed{5}\ \boxed{9}\ \boxed{12}\ \boxed{1}\ \boxed{13}\ \boxed{10}\ \boxed{2}\ \boxed{3}$$

A 4082

B 4095

C 4096

D 8178

E 8191

Solution(s):

Let n be the number of cards picked up on the first pass, where $1 \leq n \leq 12$.

If we choose the spaces that the n cards occupy, the positions of the remaining cards are determined since they must be placed in order.

There are $\binom{13}{n}$ ways to choose where the n cards go, but if the n cards are placed at the very beginning, then all the cards will be picked up on the first pass.

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{10}\ \boxed{11}\ \boxed{12}\ \boxed{13}$$

Therefore, for a given n there are $\binom{13}{n} - 1$ ways to arrange the cards.

We need to now find the sum over all possible n, which equals

$$\sum_{i=1}^{12} \binom{13}{n} - 1 =$$

$$\left(\sum_{i=0}^{13} \binom{13}{n}\right) - 14 = 2^{13} - 14$$

$$= 8192 - 14 = 8178.$$

Thus, **D** is the correct answer.

23. Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3,$ and $PD = 4$. What is $\frac{BC}{AD}$?

A $\frac{1}{4}$

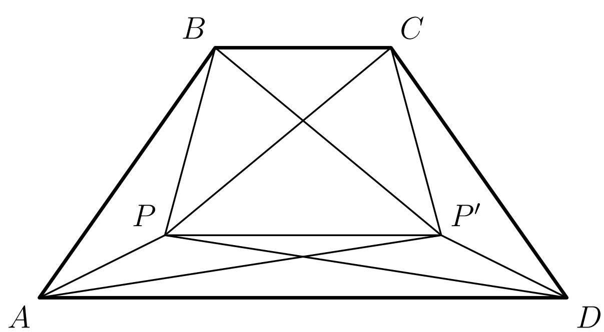
B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution(s):



Let P' be the reflection of P across the perpendicular bisector of $\overline{BC}$.

This forms two new isosceles trapezoids: $CBPP'$ and $DAPP'$.

Therefore, we get \begin{gather*} P'A = PD = 4 \\ P'D = PA = 1 \\ P'C = PB = 2 \\ P'B = PC = 3. \end{gather*}

Using Ptolemy's theorem, we know that the product of the diagonals is equal to the sum of the products of the opposite sides. Therefore: \begin{gather*} PP' \cdot AD + 1 = 16 \\ PP' \cdot BC + 4 = 9. \end{gather*}

This gets us $PP' \cdot AD = 15$ and $PP' \cdot BC = 5$. Dividing these two equations yields $\dfrac{BC}{AD} = \dfrac{1}{3}$.

Thus, **B** is the correct answer.

24. How many strings of length 5 formed from the digits 0, 1, 2, 3, 4, are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j?

(For example, 02214 satisfies this condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

A 500

B 625

C 1089

D 1199

E 1296

Solution(s):

Note that there must be at least one 0 to satisfy the condition. We can proceed by casework on the number of distinct digits in the string.

1 **digit**

The only possible digit is just 0. This can be done in 1 way.

2 **digits**

0,1	0,2	0,3	0,4	Arrangements
00001	00002	00003	00004	$4 \cdot \frac{5!}{4! \cdot 1!}$
00011	00022	00033		$3 \cdot \frac{5!}{3! \cdot 2!}$
00111	00222			$2 \cdot \frac{5!}{2! \cdot 3!}$
01111				$1 \cdot \frac{5!}{1! \cdot 4!}$

This gives us a total of

$$20 + 30 + 20 + 5 = 75$$

strings.

3 **digits**

0,1,2	0,1,3	0,1,4	0,2,3	0,2,4	0,3,4	Arrangements
00012	00013	00014	00023	00024	00034	$6 \cdot \frac{5!}{3! \cdot 1! \cdot 1!}$
00112	00113	00114	00223	00224		$5 \cdot \frac{5!}{2! \cdot 2! \cdot 1!}$
00122	00133		00233			$3 \cdot \frac{5!}{2! \cdot 1! \cdot 2!}$
01112	01113	01114				$3 \cdot \frac{5!}{1! \cdot 3! \cdot 1!}$
01122	01133					$2 \cdot \frac{5!}{1! \cdot 2! \cdot 2!}$
01222						$1 \cdot \frac{5!}{1! \cdot 1! \cdot 3!}$

This gives us a total of

$$120 + 150 + 90 + 60 + 60$$

$$+20 = 500$$

strings.

4 **digits**

Digits	0,1,2,3	0,1,2,4	0,1,3,4	0,2,3,4
	00123	00124	00134	00234
	01123	01124	01134	
	01223	01224		
	01233			

This gives us a total of

$$10 \cdot \frac{5!}{2! \cdot 1! \cdot 1! \cdot 1!} = 600$$

strings.

5 **digits**

This gives us $5! = 120$ strings.

All together, we have

$$1 + 75 + 500 + 600$$

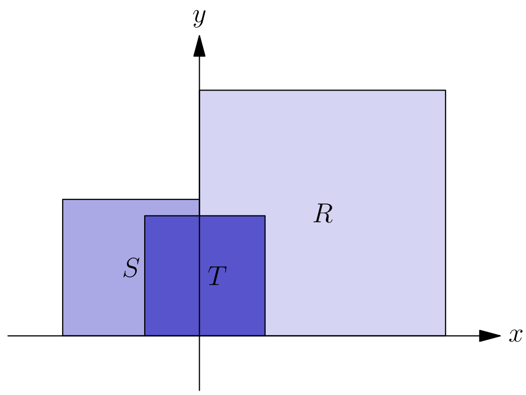
$$+120 = 1296$$

strings.

Thus, **E** is the correct answer.

25. Let R, S, and T be squares that have vertices at lattice points (i.e., points whose coordinates are both integers) in the coordinate plane, together with their interiors.

The bottom edge of each square is on the x-axis. The left edge of R and the right edge of S are on the y-axis, and R contains $\frac{9}{4}$ as many lattice points as does S. The top two vertices of T are in $R \cup S$, and T contains $\frac{1}{4}$ of the lattice points contained in $R \cup S$. See the figure (not drawn to scale).



The fraction of lattice points in S that are in $S \cap T$ is 27 times the fraction of lattice points in R that are in $R \cap T$. What is the minimum possible value of the edge length of R plus the edge length of S plus the edge length of T?

A 336

B 337

C 338

D 339

E 340

Solution(s):

Let r be the number of lattice points on the side length of R. Similarly define s for S and t for T. Note that the number of lattice points in a rectangle is the product of the number of lattice points along its width and the number of lattice points along its length.

The first conditions gives us that

$$r^2 = \frac{9}{4} \cdot s^2$$
$$r = \frac{3}{2} \cdot s \tag{1}$$

The number of lattice points in $R \cup T$ is the sum of the lattice points in each of the regions, but there is overlap along the y-axis where S touches it.

The second condition, therefore, yields

$$t^2 = \frac{1}{4}(r^2 + s^2 - s)$$
$$t^2 = \frac{1}{4}(\frac{9}{4} \cdot s^2 + s^2 - s)$$
$$t^2 = \frac{1}{4} \cdot \frac{13s^2 - 4s}{4}$$
$$16t^2 = s(13s - 4).$$

From (1), we get that s is a multiple of 2. We can substitute s with $2j$ to get

$$16t^2 = 2j(26j - 4)$$
$$4t^2 = j(13j - 2).$$

For the product to be divisible by 4, j must be divisible by 2. We can again substitute j with $2k$ to get

$$4t^2 = 2k(26k - 2)$$
$$t^2 = k(13k - 1) \tag{2}$$

Let x be the number of lattice points along the bottom of the rectangle formed by $S \cap T$ and y be the number of lattice points along the bottom of the rectangle formed by $R \cap T$.

Using these variable, we get that the number of lattice points in $S \cap T$ is xt and in $R \cap T$ is yt.

The third condition gives us that

$$\frac{xt}{s^2} = 27 \cdot \frac{yt}{r^2}$$

$$\frac{x}{s^2} = 27 \cdot \frac{y}{\frac{9}{4}s^2} x = 12y.$$

We also know that $t = x + y - 1$ (accounting for overlap), and this yields

$$t = 13y - 1 \tag{3}$$

(3) gives us that

$$t \equiv -1 \bmod 13 t^2 \equiv 1 \bmod 13.$$

However, by (2), we get that

$$t^2 \equiv k \cdot -1 \bmod 13$$

$$k \equiv -1 \bmod 13.$$

By (2), we also get that k is a perfect square since it is relatively prime to $13k - 1$, and they must multiply to a perfect square.

Using these restrictions on k, we can try to find the smallest k that works. We get that $k = 25$ satisfies both conditions.

From this value of k, we get that $j = 2 \cdot 25 = 50$, $s = 2 \cdot 50 = 100$, and $r = \frac{3}{2} \cdot 100 = 150$. We can also find that

$$t^2 = 25(13 \cdot 25 - 1) = 25 \cdot 324$$

$$t = 5 \cdot 18 = 90.$$

Therefore,

$$r + s + t = 340.$$

The question, however, asked for the sum of the side lengths. The side lengths of the squares are 1 less than the number of lattice points on the side, so we have to subtract 3.

Therefore, the desired answer is $340 - 3 = 337$.

Thus, **B** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2022A