2022 AMC 10B Solutions

Typeset by: LIVE, by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2022B/solutions

1. Define $x \diamondsuit y$ to be $|x-y|$ for all real numbers x and y. What is the value of

$$(1 \diamondsuit (2 \diamondsuit 3)) - ((1 \diamondsuit 2) \diamondsuit 3)?$$

A -2

B -1

C 0

D 1

E 2

Solution(s):

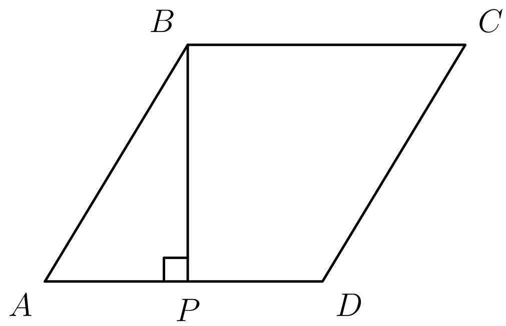
$$(1 \diamondsuit (2 \diamondsuit 3)) - ((1 \diamondsuit 2) \diamondsuit 3) =$$

$$|1 - |2-3|| - ||1-2|-3| =$$

$$|1-1| - |1-3| = 0 - 2 = -2.$$

Thus, the answer is **A**.

2. In rhombus $ABCD$, point P lies on segment $\overline{AD}$ so that $\overline{BP} \perp \overline{AD}$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$? (Note: The figure is not drawn to scale.)



A $3\sqrt{5}$

B 10

C $6\sqrt{5}$

D 20

E 25

Solution(s):

Since we have a rhombus, we know

$$AB = AD = AP + PD = 5.$$

And by the Pythagorean Theorem,

$$AP^2 + BP^2 = AB^2$$

We know that

$$9 + BP^2 = 25.$$

$$BP = 4.$$

The area of a rhombus is bh, so the area is $5 \cdot 4 = 20$.

Thus, the answer is **D**.

3. How many three-digit positive integers have an odd number of even digits?

A 150

B 250

C 350

D 450

E 550

Solution(s):

First, we can choose any combination for the first two digits. This would have $9 \cdot 10 = 90$ choices.

Then, if there are an odd number of even digits among them, I make the units digit odd, which can be done in 5 ways. Otherwise, I make the units digit even, which can be done in 5 ways. Regardless of my choice of the first two digits, I have 5 ways to choose the units digit.

Therefore, there are 90 ways to choose the first two digits, and 5 ways to choose the last two, so the total number of ways is $90 \cdot 5 = 450$.

Thus, the answer is **D**.

4. A donkey suffers an attack of hiccups and the first hiccup happens at $4:00$ one afternoon. Suppose that the donkey hiccups regularly every 5 seconds. At what time does the donkey's 700th hiccup occur?

- A 15 seconds after $4:58$
- B 20 seconds after $4:58$
- C 25 seconds after $4:58$
- D 30 seconds after $4:58$
- E 35 seconds after $4:58$

Solution(s):

Since we want to look at the 700th hiccup, we need to look at time that is 699 hiccups after the first one.

This would be $699 \cdot 5 = 3495$ seconds. Note that $3495 = 60 \cdot 58 + 15$, so the time would be 58 minutes and 15 seconds after the first hiccup. This would therefore be $4:58$ and 15 seconds.

Thus, the answer is **A**.

5. What is the value of

$$\frac{(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})}{\sqrt{(1-\frac{1}{3^2})(1-\frac{1}{5^2})(1-\frac{1}{7^2})}}?$$

A $\sqrt{3}$

B 2

C $\sqrt{15}$

D 4

E $\sqrt{105}$

Solution(s):

Lets work with the denominator first. By using difference of two squares on each term, we get the denominator as

$$\sqrt{(1-\frac{1}{3^2})(1-\frac{1}{5^2})(1-\frac{1}{7^2})}$$

$$=\sqrt{(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})}$$

$$\cdot\sqrt{(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})}$$

$$=\sqrt{(\frac{2}{3})(\frac{4}{5})(\frac{6}{7})}\sqrt{(\frac{4}{3})(\frac{6}{5})(\frac{8}{7})}.$$

Furthermore, we simplify the numerator as follows:

$$(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})$$

$$=(\frac{4}{3})(\frac{6}{5})(\frac{8}{7}).$$

Dividing the numerator and denominator yields

$$\begin{aligned}
&\frac{(\frac{4}{3})(\frac{6}{5})(\frac{8}{7})}{\sqrt{(\frac{2}{3})(\frac{4}{5})(\frac{6}{7})}\sqrt{(\frac{4}{3})(\frac{6}{5})(\frac{8}{7})}} \\
&= \frac{\sqrt{(\frac{4}{3})(\frac{6}{5})(\frac{8}{7})}}{\sqrt{(\frac{2}{3})(\frac{4}{5})(\frac{6}{7})}} \\
&= \frac{\sqrt{8}}{\sqrt{2}} \\
&= 2.
\end{aligned}$$

Thus, the answer is **B**.

6. How many of the first ten numbers of the sequence

$$121, 11211, 1112111, \ldots$$

are prime numbers?

A 0

B 1

C 2

D 3

E 4

Solution(s):

We claim that none of these numbers can ever be prime.

We prove this claim by noticing that the nth number is

$$\sum_{k=0}^{2n} 10^k + 10^n = \sum_{k=0}^{n} 10^k +$$

$$\sum_{k=n}^{2n} 10^k = \sum_{k=0}^{n} 10^k + \sum_{k=0}^{n} 10^k \cdot 10^n$$

$$= (10^n + 1)(\sum_{k=0}^{n} 10^k).$$

This shows that the number can be written as the product of two numbers greater than 1, so there are no primes.

Thus, the answer is **A**.

7. For how many values of the constant k will the polynomial $x^2 + kx + 36$ have two distinct integer roots?

A 6

B 8

C 9

D 14

E 16

Solution(s):

Let the roots be r, s. Then:

$$\begin{aligned} & x^2 + kx + 36 \\ & = (x - r)(x - s) \\ & = x^2 - (r + s)x + rs \\ & = 0 \end{aligned}$$

And so, $rs = 36$ and $r + s = -k$.

Therefore, we need r and s distinct such that $rs = 36$. All the possible factor pairs are

$$\pm\{1, 36\}, \pm\{2, 18\}, \pm\{3, 12\}$$
$$\text{and } \pm\{4, 9\}.$$

Each of these unordered pairs produces a unique value for k, so there are 8 possible values for k.

Thus, **B** is the correct answer.

8. Consider the following 100 sets of 10 elements each:

$$\{1,2,3,\ldots,10\},$$
$$\{11,12,13,\ldots,20\},$$
$$\{21,22,23,\ldots,30\},$$
$$\vdots$$
$$\{991,992,993,\ldots,1000\}.$$

How many of these sets contain exactly two multiples of 7?

- A 40
- B 42
- C 43
- D 49
- E 50

Solution(s):

We can analyze the units digit of the first multiple of 7 in each set.

If the last digit is $4,5,6,7$, then adding 7 yields numbers outside the set, so the sets with a multiple of 7 such that its units digit is $4,5,6,7$ would have only one multiple.

If the last digit is $1,2,3$ then adding 7 would yield a units digit of $8,9,0$ in the same set.

Out of the first 98 sets, there are an equal number of occurrences of sets such that the first multiple of 7 has each of the units digit of $1,2,3,4,5,6,7$ since they cycle every 7 sets. These yield 42 sets whose first multiple of 7 contains two multiples of 7.

The 99th and 100th set wouldn't work since they yield a set whose first multiples are 987 and 994 respectively, which can't have 2 multiples of 7. This means we have 42 sets that work.

Thus, the answer is **B**.

9. The sum

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{2021}{2022!}$$

can be expressed as $a-\frac{1}{b!}$, where a and b are positive integers. What is $a+b$?

A 2020

B 2021

C 2022

D 2023

E 2024

Solution(s):

We claim

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n-1}{n!}$$
$$=1-\frac{1}{n!}.$$

To prove this, we can use induction.

If $n=2$, then the sum is $\frac{1}{2!}=1-\frac{1}{2!}$.

If it works for $n-1$, then

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n-1}{n!}$$
$$=1-\frac{1}{(n-1)!}+\frac{n-1}{n!}$$
$$=1-\frac{n}{n!}+\frac{n-1}{n!}=1-\frac{1}{n!}.$$

This means our formula is proven, so we can get our answer by plugging in $n=2022$. Our answer is $1-\frac{1}{2022!}$, yielding $a=1, b=2022$, making our answer 2023.

Thus, our answer is **D**.

10. Camila writes down five positive integers. The unique mode of these integers is 2 greater than their median, and the median is 2 greater than their arithmetic mean. What is the least possible value for the mode?

A 5

B 7

C 9

D 11

E 13

Solution(s):

Let the integers in order be a, b, c, d, e.

The median of this list is c. Since the mode is greater than the median, both d and e are equal to the mode, so we can write the list as $a, b, c, c+2, c+2$.

The mean is now

$$\frac{a+b+c+c++c+2}{5}$$

$$=c-2,$$

which yields

$$a+b+3c+4=5c-10.$$

This means

$$a+b=2c-14.$$

This means $a+b$ is even, and a, b are different positive integers since we have a unique mode. This means $a+b \geq 4$, since the uniqueness eliminates $a=1, b=1$, so $a+b \neq 2$. This means $a+b=4$ yielding a median of $c=9$, making the mode 11.

Thus, the answer is **D**.

11. All the high schools in a large school district are involved in a fundraiser selling T-shirts. Which of the choices below is logically equivalent to the statement "No school bigger than Euclid HS sold more T-shirts than Euclid HS"?

- A All schools smaller than Euclid HS sold fewer T-shirts than Euclid HS.
- B No school that sold more T-shirts than Euclid HS is bigger than Euclid HS.
- C All schools bigger than Euclid HS sold fewer T-shirts than Euclid HS.
- D All schools that sold fewer T-shirts than Euclid HS are smaller than Euclid HS.
- E All schools smaller than Euclid HS sold more T-shirts than Euclid HS.

Solution(s):

First, we have no information about schools that are smaller than Euclid HS, so we can eliminate all the choices that mention smaller schools. This leaves just **B** and **C** to look at.

Now, given our statement, we know that if a school is bigger than Euclid, then it couldn't have sold more than Euclid. This means if a school sold more, it couldn't have been bigger, which corresponds to choice **B**.

Thus, the answer is **B**.

12. A pair of fair 6-sided dice is rolled n times. What is the least value of n such that the probability that the sum of the numbers face up on a roll equals 7 at least once is greater than $\frac{1}{2}$?

- A 2
- B 3
- C 4
- D 5
- E 6

Solution(s):

To compute this, we can also find the least n such that the probability of not rolling a 7 is less than $\frac{1}{2}$. Each roll has an independent probability of $\frac{1}{6}$ of getting 7, so it has a $\frac{5}{6}$ probability of not landing on 7.

Thus, the probability of none of the rolls being 7 is $(\frac{5}{6})^n$. We must find the least n such that $(\frac{5}{6})^n < \frac{1}{2}$.

If $n = 3$, then the probability is $\frac{125}{216}$, which is greater than $\frac{1}{2}$.

If $n = 4$, then the probability is $\frac{625}{1296}$, which is less than $\frac{1}{2}$. This makes the answer 4.

Thus, the answer is **C**.

13. The positive difference between a pair of primes is equal to 2, and the positive difference between the cubes of the two primes is 31106. What is the sum of the digits of the least prime that is greater than those two primes?

A 8

B 10

C 11

D 13

E 16

Solution(s):

Since the primes are 2 away from each other, we can make them equal to $m - 1, m + 1$, where m is their average.

Then,

$$(m+1)^3 - (m-1)^3 = 31106,$$

making

$$m^3 + 3m^2 + 3m + 1$$

$$-(m^3 - 3m^2 + 3m - 1)$$

$$= 6m^2 + 2 = 31106.$$

Therefore, $m^2 = 5184$, so $m = 72$.

The primes are therefore $71, 73$. The least prime greater than both of those is 79, and its digit sum is 16.

Thus, the answer is **E**.

14. Suppose that S is a subset of $\{1, 2, 3, \cdots, 25\}$ such that the sum of any two (not necessarily distinct) elements of S is never an element of S. What is the maximum number of elements S may contain?

- A 12
- B 13
- C 14
- D 15
- E 16

Solution(s):

First, note that we can make a set with size 13 using $S = \{13, 14 \cdots, 25\}$.

Now, we prove no arbitrary set of size greater than 13 work. Let m be the maximum element of S. Then, for all i in S, we know $m - i$ isn't in S.

This would eliminate $\lceil \frac{m-1}{2} \rceil$ of the numbers below m. This means the maximum number of elements below m is $\lfloor \frac{m-1}{2} \rfloor$, making the maximum number of elements $\lfloor \frac{m-1}{2} \rfloor + 1$.

The maximum value of this has $m = 25$, yielding 13.

Thus, the answer is **B**.

15. Let S_n be the sum of the first n term of an arithmetic sequence that has a common difference of 2. The quotient $\frac{S_{3n}}{S_n}$ does not depend on n. What is S_{20}?

A 340

B 360

C 380

D 400

E 420

Solution(s):

Let the sequence be $a_1, a_2, \cdots$. Create a by making it the term before a_1 in the sequence. This would make $a_n = a + 2n$.

This would make

$$S_n = \sum_{i=1}^{n} (a + 2i) =$$

$$a \cdot n + 2 \sum_{i=1}^{n} i = an + n^2 + n.$$

This makes

$$\frac{S_{3n}}{S_n} = \frac{3an + 9n^2 + 3n}{an + n^2 + n} =$$

$$\frac{3a + 9n + 3}{a + n + 1}.$$

Thus, we must find a such that this value is constant.

If our given value is constant, than the given value minus 9 is constant, so

$$\frac{3a + 9n + 3}{a + n + 1} - 9 = \frac{-6a - 6}{a + n + 1}$$

is constant. As n increases, the numerator is constant and the denominator is increasing. Therefore, if the number is constant, the numerator must be 0.

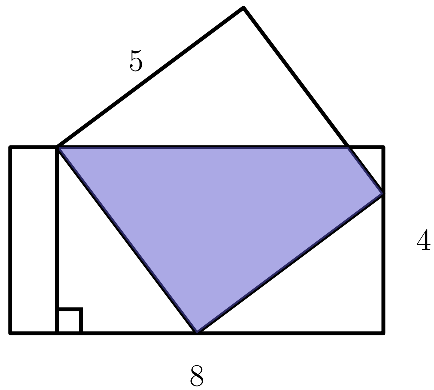
Since $-6a - 6 = 0$, we have $a = -1$.

With our formula from before, we have

$$S_{20} = -1(20) + 20^2 + 20 = 400.$$

Thus, the answer is **D**.

16. The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



A $15\frac{1}{8}$

B $15\frac{3}{8}$

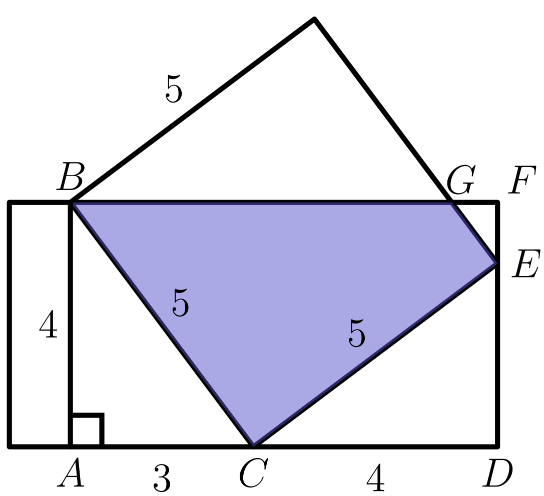
C $15\frac{1}{2}$

D $15\frac{5}{8}$

E $15\frac{7}{8}$

Solution(s):

Firstly, let's label the points as follows:



Since we have a rectangle, $AB = 4$. By the Pythagorean Theorem, we have $AC = 3$. Then, since $\angle ACB$ and $\angle DCE$ are complementary, $\angle BAC = \angle CDE = 90°$, and $BC = CE$ we know $ABC \cong CDE$. Therefore, $ED = 3$ and $EF = 1$.

Since $\angle CED$ and $\angle GEF$ are complementary and $\angle GEF = \angle CDE = 90°$, we know EFG and CDE are similar. This means $\frac{EC}{CD} = \frac{EG}{EF}$, so $EG = 1.25$. Since the shaded region is a trapezoid, we can get the area as

$$\begin{aligned}\frac{(GE + BC)EC}{2} &= \frac{5(5 + 1.25)}{2} \\ &= \frac{5 \cdot 6.25}{2} = 15.625.\end{aligned}$$

This is equal to $15\frac{5}{8}$.

Thus, the answer is **D**.

17. One of the following numbers is not divisible by any prime number less than 10. Which is it?

- A $2^{606} - 1$
- B $2^{606} + 1$
- C $2^{607} - 1$
- D $2^{607} + 1$
- E $2^{607} + 3^{607}$

Solution(s):

Note that $a^n - b^n$ is divisible by $a - b$.

For **A**, let $a = 4, b = 1, n = 303$. Then we get that

$$4^{30}3 - 1 = 2^{606} - 1$$

is divisible by 3.

For **B**, let $a = 4, b = -1, n = 303$. Then we get that

$$4^{303} - (-1)^{303} = 2^{606} - 1$$

is divisible by 5.

For **D**, since $2^{606} - 1$ is divisible by 3, we have $2^{607} - 2$ is divisible by 3, leaving $2^{607} + 1$ being divisible by 3.

For **E**, let $a = 3, b = -2, n = 607$. Then we get that

$$3^{607} - (-2)^{607} = 3^{607} + 2^{607}$$

is divisible by 5.

For **C**, we know $2^{607} + 1$ is divisible by 3, so our value isn't divisible by 3. We also know $2^{607} + 1$ is divisible by 5 from choice D so our value isn't divisible by 5.

Also, $64^{101} - 1 = 2^{606} - 1$ is divisible by 7, so $2^{607} - 2$ is divisible by 7. This means our value isn't divisible by 7. Since it isn't divisible by 2, it isn't divisible by a prime under 10.

Thus, our answer is **C**.

18. Consider systems of three linear equations with unknowns x, y, and z,

$$\begin{cases} a_1x+b_1y+c_1z &= 0 \\ a_2x+b_2y+c_2z &= 0 \\ a_3x+b_3y+c_3z &= 0 \end{cases}$$

where each of the coefficients is either 0 or 1 and the system has a solution other than $x=y=z=0$. For example, one such system is

$$\begin{cases} 1x+1y+0z &= 0 \\ 0x+1y+1z &= 0 \\ 0x+0y+0z &= 0 \end{cases}$$

with a nonzero solution of $(x,y,z)=(1,-1,1)$. How many such systems of equations are there? (The equations in a system need not be distinct, and two systems containing the same equations in a different order are considered different.)

A 302

B 338

C 340

D 343

E 344

Solution(s):

There are $2^9=512$ total configurations. Now, we can use complementary counting to determine how many have more than one solution.

If a configuration has 3 equations which don't contain redundant information, then it has only one solution.

This means every equation has to be different. Also, if any equation has $a,b,c=0$, then it doesn't provide any information, making it redundant. This means we have 7 choices for the first equation, 6 choices for the second, and 5 choices for the third.

This yields 210 configurations. However, some configurations may still yield redundant information. If two equations add to the other equation, then there is a redundancy.

There are two cases for this to happen.

Case $1:1$ of the equations has $a, b, c = 1$, another equation has 1 of the variables being 1 and the other equation has 2 variables being 1. There are 3 ways to choose which equation has every variable as 1. Then, there are 2 ways to choose which variables have one variable being 1, and this equation has 3 ways to choose which variable is 1. This case has $3 \cdot 2 \cdot 3 = 18$ configurations to exclude.

Case $2:1$ of the equations has 2 variables being 1, and the other two equations have only one variable being 1, with those variables being different from each other, but one of the variables chosen in the first equation. There are 3 ways to choose the equation with 2 variables being 1, there are 3 ways to choose which variables are 1, and 2 ways to choose the order of the other equations. This case has $3 \cdot 2 \cdot 3 = 18$ configurations to exclude.

There are a total of $210 - 18 - 18 = 174$ cases which have only one solution. This means $512 - 174 = 338$ configurations have multiple solutions, making at least one nonzero.

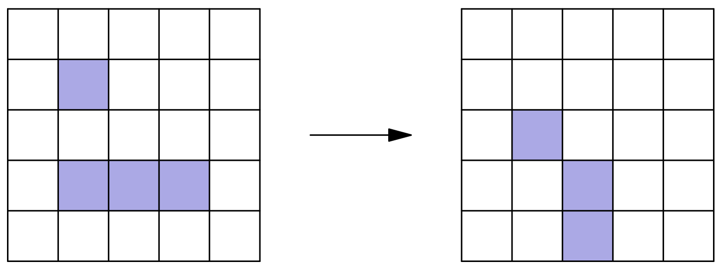
Thus, the answer is **B**.

19. Each square in a 5×5 grid is either filled or empty, and has up to eight adjacent neighboring squares, where neighboring squares share either a side or a corner. The grid is transformed by the following rules:

- Any filled square with two or three filled neighbors remains filled.

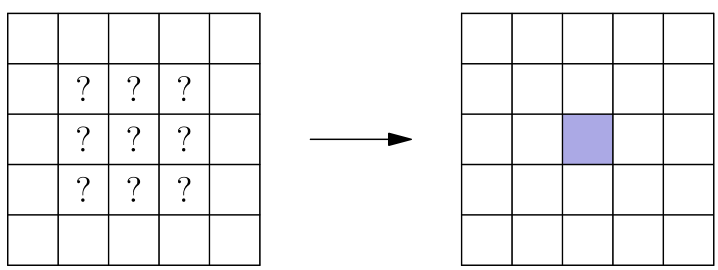
- Any empty square with exactly three filled neighbors becomes a filled square.

- All other squares remain empty or become empty. A sample transformation is shown in the figure below.



Initial Transformed

Suppose the 5×5 grid has a border of empty squares surrounding a 3×3 subgrid. How many initial configurations will lead to a transformed grid consisting of a single filled square in the center after a single transformation? (Rotations and reflections of the same configuration are considered different.)



Initial Transformed

A 14

B 18

C 22

D 26

E 30

Solution(s):

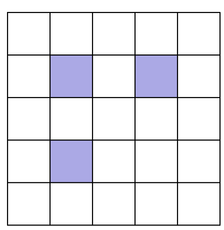
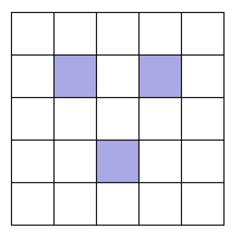
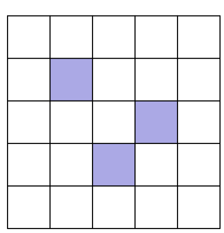
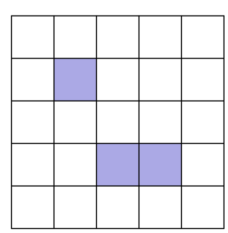
Suppose the center is initially filled. Then, there are either either 2 or 3 other filled squares, each of which can't have 2 or 3 filled neighbors.

This means that there are at most 4 filled squares, so each square has at most 3 neighbors. Since they don't have 2 or 3 neighbors, they must have at most 1 neighbor. The center square is a neighbor, so they can't have any other neighbor.

Suppose I have a filled square on an edge. Since there is some filled square that isn't a neighbor of the square, we can examine the two edges which are neighbors of the filled edge. If I have a filled edge on the corner, the edge on the same side as the corner would have three neighbors. If I choose the opposite edge, the adjacent edges would have three neighbors.

Suppose I choose a corner. Then, I need to choose another corner. If I choose the adjacent corner, then the edge between would have three neighbors, making it filled. Therefore, it must be the adjacent corner. This has 2 configurations.

Suppose the center is initially empty. Then, there are 3 filled neighbors of the center, each with at most 1 neighbors. This means no square has two filled neighbors. This makes it only possible to do in the following ways:

The first three can rotated making 4 configurations, and the last one can be rotated and reflected making 8 configurations. There are 20 configurations with the center being empty. This means there are $20 + 2 = 22$ different configurations.

Thus, the answer is **C**.

20. Let $ABCD$ be a rhombus with $\angle ADC = 46^\circ$. Let E be the midpoint of $\overline{CD}$, and let F be the point on $\overline{BE}$ such that $\overline{AF}$ is perpendicular to $\overline{BE}$. What is the degree measure of $\angle BFC$?

A 110

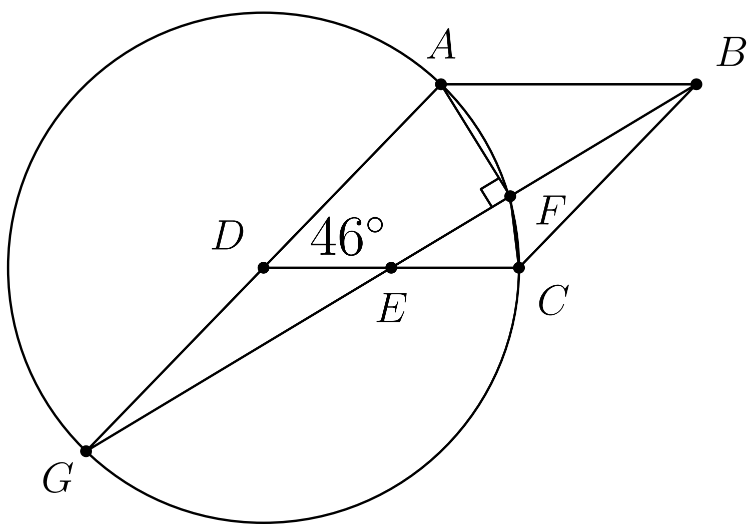
B 111

C 112

D 113

E 114

Solution(s):



First, we extend BE and AD such that they meet at G. Since

$$\angle GDE = \angle ECB,$$

$$\angle GED = \angle BEC \text{ and } DE = EC,$$

we know $GDE \cong BCE$. Therefore, $DG = BC = AD$. This means that if we construct a circle with center D that includes A, C, G are also on it.

Also, since AFG is a right triangle, the drawn circle would be its circumcircle, placing F on the circle.

Since $\angle GDC = 134^\circ$, we can get

$$\angle CFE = \angle CFG = \frac{\overset{\frown}{CG}}{2}$$

$$= \frac{134^\circ}{2} = 67^\circ.$$

Therefore,

$$\angle BFC = 180^\circ - 67^\circ = 113^\circ.$$

Thus, the answer is **D**.

21. Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2 + x + 1$, the remainder is $x + 2$, and when $P(x)$ is divided by the polynomial $x^2 + 1$, the remainder is $2x + 1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

A 10

B 13

C 19

D 20

E 23

Solution(s):

Since $P(x)$ has a remainder of $x + 2$ when divided by $x^2 + x + 1$, it must be able to be written as

$$P(x) =$$

$$(x^2 + x + 1)Q(x) + x + 2$$

for some polynomial $Q(x)$. Note that the remainder of $P(x)$ when divided by $x^2 + 1$ is equal to the remainder when $xQ(x) + x + 2$.

If $Q(x) = c$ for some constant, then the remainder when $P(x)$ is $(c + 1)x + 2$, which can't happen since we need a 1 for our constant.

If $P(x) = ax + b$, then the remainder when divided by x^1 is equal to the remainder when

$$(ax + b)x + x + 2 =$$

$$ax^2 + (b + 1)x + 2$$

is divided by $x^2 + 1$. We can get the remainder by subtracting $a(x^2 + 1)$, yielding $(b + 1)x + (2 - a)$. This means $b + 1 = 2$ and $2 - a = 1$, so $a = b = 1$.

This means

$$P(x) =$$

$$(x+1)(x^2+x+1)+x+2=$$

$$x^3+2x^2+3x+3.$$

The sum of the squares of the coefficients is 23.

Thus, the answer is **E**.

22. Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations

$$x^2 + y^2 = 4,$$
$$x^2 + y^2 = 64,$$
$$(x-5)^2 + y^2 = 3.$$

What is the sum of the areas of all circles in S?

A 48π

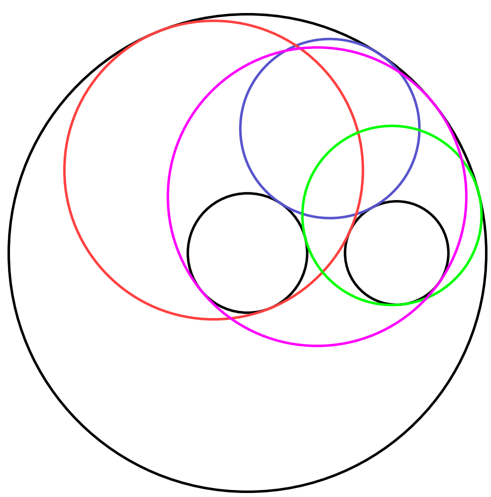
B 68π

C 96π

D 102π

E 136π

Solution(s):



Let $x^2 + y^2 = 64$ be circle O, $x^2 + y^2 = 5$ be circle P, and $(x-5)^2 + y^2 = 3$ be circle Q.

First note that every circle, R, in S is internally tangent to O. Then we case on the tangency of R with P and Q.

Case 1 : This corresponds to the pink circle. This is where P and Q are internally tangent to R.

Case 2 : This corresponds to the bluish circle. This is where P and Q are externally tangent to R.

Case 3 : This corresponds to the green circle. This is where P is externally and Q is internally tangent to R.

Case 4 : This corresponds to the red circle. This is where P is internally and Q is externally tangent to R.

We can consider cases 1 and 4 together. Note that O and P have the same center. This means that the line connecting the center of R and O passes through the tangency point of both S and O and S and P.

This line is the diameter of R, and it has length

$$r_P + r_O = 2 + 8 = 10.$$

Therefore, the radius of R is 5.

Consider cases 2 and 3 together. Similarly to above, the line connecting the center of R and O will pass through the tangency points.

This time, however, the diameter of R is

$$r_P - r_O = 8 - 2 = 6.$$

This makes the radius of R 3.

S contains 8 circles: 4 of which have radius 5 and 4 of which have radius 3 (this is because we can flip all the circles in the diagram over the x-axis to get 4 more circles).

The total area of the circles in S is therefore

$$4(5^2\pi + 3^2\pi) = 136\pi.$$

Thus, **E** is the correct answer.

23. Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n = 1, 2, 3,$ Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0, 1)$. During the nth step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the nth step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

A $\frac{1}{3}$

B $\frac{1}{2}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{5}{6}$

Solution(s):

We begin by breaking the question into two smaller problems:

- The probability such that if x, y were random numbers in the interval $(0, 1)$, that $x + y < 1$.

- The probability such that if x, y, z were random numbers in the interval $(0, 1)$, that $x + y + z < 1$.

To solve both of these, we use geometric probability. That means, for a 2-D or 3-D coordinate system, that $x + y < 1$ and $x + y + z < 1$.

To solve the first one, we take the area of the region where $x + y < 1$, and $x, y > 0$. This would be $\frac{1}{2}$ since it is a triangle with base and height of 1. Since the total area is just $1^2 = 1$, the probability is also $\frac{1}{2}$.

To solve the second one, we take the area of the region where $x + y + z < 1$, and $x, y > 0$. This would be $\frac{1}{6}$ since it is a volume of a pyramid with base of area $\frac{1}{2}$ and height of 1. Again, the volume is $1^3 = 1$, so the probability is also $\frac{1}{2}$.

Now, we can look at the cases of the problem. Annie can either go until $n = 2$ if $t_1 + t_2 > 1$ or until $n = 3$ otherwise.

Case 1 : The probability that $t_1 + t_2 > 1$ is $\frac{1}{2}$ since the probability that $t_1 + t_2 < 1$ is $\frac{1}{2}$. Similarly, the probability that $x_1 + x_2 > 1$ is $\frac{1}{2}$. The total probability with this case is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Case 2 : The probability that $t_1 + t_2 < 1$ is $\frac{1}{2}$. Also, the probability that $x_1 + x_2 + x_3 > 1$ is $\frac{5}{6}$ since the probability that $x_1 + x_2 + x_3 < 1$ is $\frac{1}{6}$.The total probability with this case is $\frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}$.

The total probability therefore is $\frac{1}{4} + \frac{5}{12} = \frac{2}{3}$.

Thus, the answer is **C**.

24. Consider functions f that satisfy

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|$$

for all real numbers x and y. Of all such functions that also satisfy the equation $f(300) = f(900)$, what is the greatest possible value of

$$f(f(800)) - f(f(400))?$$

A 25

B 50

C 100

D 150

E 200

Solution(s):

Note that

$$|f(f(400)) - f(f(300))|$$
$$\leq \frac{1}{2}|f(400) - f(300)|$$
$$\leq \frac{1}{4}|400 - 300| = 25,$$

and

$$|f(f(900)) - f(f(800))|$$
$$\leq \frac{1}{2}|f(900) - f(800)|$$
$$\leq \frac{1}{4}|900 - 800| = 25.$$

Since $f(900) = f(300)$, by the triangle inequality, we know

$$|f(f(800)) - f(f(400))| =$$

$$|(f(f(800)) - f(f(900)))-$$
$$(f(f(400)) - f(f(300)))| \leq$$
$$|(f(f(800)) - f(f(900)))| +$$
$$|(f(f(400)) - f(f(300)))|$$
$$\leq 50.$$

Now, we must conclude this value is attainable. We can make $f(x)$ a piecewise function such that $f(x) = 600$ if $x > 900$ or $x < 300$,

$$f(x) = -\frac{1}{2}(x - 300) + 600$$

if $300 \leq x \leq 400$,

$$f(x) = \frac{1}{2}(x - 600) + 600$$

if $400 < x < 800$, and

$$f(x) = -\frac{1}{2}(x - 900) + 600$$

if $800 \leq x \leq 900$. This would make $f(f(400)) = f(550) = 575$ and $f(f(800)) = f(650) = 625$. This yields a difference of 50, so our result holds.

Thus, the answer is **B**.

25. Let $x_0, x_1, x_2, \ldots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} +$$

$$4x_{2021} + 8x_{2022}?$$

A 6

B 7

C 12

D 14

E 15

Solution(s):

Note first that

$$\frac{S_{2023} - S_{2019}}{2^{2019}} = x_{2019} +$$

$$2x_{2020} + 4x_{2021} + 8x_{2022}.$$

Therefore, we should attempt to find S_{2019}, S_{2023}. Also, note that

$$0 \leq S_n \leq \sum_{k=0}^{n-1} 2^k < 2^n.$$

Now, since $7S_n \equiv 1 \pmod{2^n}$, we know

$$7S_n = m2^n + 1 \implies$$

$$S_n = \frac{m2^n + 1}{7}$$

for some integer m. Also, since $0 \le S_n < 2^n$, we know

$$0 \le \frac{m2^n + 1}{7} < 2^n.$$

This means $0 \le m < 7$. Now, we find m such that $m2^n + 1$ is divisible by 7. This makes $m2^n + 1 \equiv 0 \mod 7$.

If $n = 2019$, then

$$0 \equiv m2^{2019} + 1 \equiv$$

$$m(2^3)^{667} + 1 \equiv$$

$$m8^{667} + 1 \equiv m + 1,$$

so $m = 6$. This makes

$$S_{2019} = \frac{6(2^{2019}) + 1}{7}.$$

If $n = 2023$, then

$$0 \equiv m2^{2019} + 1$$

$$\equiv m(2^3)^{668} \cdot 2 + 1$$

$$\equiv m8^{668} \cdot 2 + 1$$

$$\equiv 2m + 1,$$

so $m = 3$. This makes

$$S_{2023} = \frac{3(2^{2023}) + 1}{7}.$$

Our answer is

$$\frac{S_{2023} - S_{2019}}{2^{2019}} =$$

$$\frac{1}{2^{2019}}\left(\frac{3(2^{2023}) + 1}{7} - \right.$$

$$\left.\frac{6(2^{2019}) + 1}{7}\right) =$$

$$\frac{1}{2^{2019}}\left(\frac{48(2^{2023})}{7}\right.$$

$$-\frac{6(2^{2019})}{7}) = \frac{42}{7} = 6.$$

Thus, the correct answer is **A**.

Problems: https://live.poshenloh.com/past-contests/amc10/2022B