2002 AMC 12B Solutions

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1. The arithmetic mean of the nine numbers in the set $\{9, 99, 999, 9999, \ldots, 999999999\}$ is a 9-digit number M, all of whose digits are distinct. The number M does not contain the digit

A 0

B 2

C 4

D 6

E 8

Solution:

Each of the nine numbers is $10^k - 1$, so their sum is $9 + 99 + \cdots + 999{,}999{,}999$. Dividing by 9,

$$\begin{aligned} M &= 1 + 11 + 111 + \cdots \\ &\quad + 111{,}111{,}111 \\ &= 123{,}456{,}789. \end{aligned}$$

Its digits are 1 through 9, so the missing digit is 0.

Thus, the correct answer is **A**.

2. What is the value of

$$(3x-2)(4x+1) \\ -(3x-2)4x+1$$

when $x=4$?

A 0

B 1

C 10

D 11

E 12

Solution:

Factor $3x-2$ out of the first two terms:

$$\begin{aligned}&(3x-2)(4x+1)\\&-(3x-2)4x+1\\&=(3x-2)(4x+1-4x)+1\\&=3x-1.\end{aligned}$$

At $x=4$ this equals $3\cdot 4-1=11.$

Thus, the correct answer is **D**.

3. For how many positive integers n is $n^2 - 3n + 2$ a prime number?

- A none
- B one
- C two
- D more than two, but finitely many
- E infinitely many

Solution:

Factor $n^2 - 3n + 2 = (n-1)(n-2)$. For $n \geq 4$ both factors exceed 1, so the value is composite. Checking $n = 1, 2, 3$ gives $0, 0,$ and 2. Only $n = 3$ yields a prime, so there is exactly one such n.

Thus, the correct answer is **B**.

4. Let n be a positive integer such that $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{n}$ is an integer. Which of the following statements is not true:

A 2 divides n

B 3 divides n

C 6 divides n

D 7 divides n

E $n > 84$

Solution:

Since $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$, the sum $\frac{41}{42} + \frac{1}{n}$ lies strictly between 0 and 2, so it must equal 1. Then $\frac{1}{n} = \frac{1}{42}$, giving $n = 42$.

Now $2, 3, 6,$ and 7 all divide 42, but $n > 84$ is false. The untrue statement is $n > 84$.

Thus, the correct answer is **E**.

5. Let $v, w, x, y,$ and z be the degree measures of the five angles of a pentagon. Suppose $v < w < x < y < z$ and v, w, x, y, z form an arithmetic sequence. Find the value of x.

A 72

B 84

C 90

D 108

E 120

Solution:

The five interior angles sum to 540°. Writing the sequence as $x - 2d, x - d, x, x + d,$ $x + 2d,$ the sum is $5x = 540,$ so $x = 108.$

Thus, the correct answer is **D**.

6. Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. Then the pair (a, b) is

A $(-2, 1)$

B $(-1, 2)$

C $(1, -2)$

D $(2, -1)$

E $(4, 4)$

Solution:

Since a and b are the roots,

$$\begin{aligned} &x^2 + ax + b \\ &= (x-a)(x-b) \\ &= x^2 - (a+b)x + ab. \end{aligned}$$

Matching coefficients gives $a + b = -a$ and $ab = b$.

As $b \neq 0$, the second equation gives $a = 1$, and then $a + b = -a$ gives $b = -2$. So $(a, b) = (1, -2)$.

Thus, the correct answer is **C**.

7. The product of three consecutive positive integers is 8 times their sum. What is the sum of their squares?

A 50

B 77

C 110

D 149

E 194

Solution:

Let the integers be $n-1, n, n+1$. Then $(n-1)n(n+1) = 8 \cdot 3n$, so $n^2 - 1 = 24$ and $n = 5$.

The three integers $4, 5, 6$ have squares summing to $16 + 25 + 36 = 77.$

Thus, the correct answer is **B**.

8. Suppose July of year N has five Mondays. Which of the following must occur five times in August of year N? (Note: Both months have 31 days.)

- A Monday
- B Tuesday
- C Wednesday
- D Thursday
- E Friday

Solution:

Since July has $31 = 4 \cdot 7 + 3$ days, the weekday of July 1 occurs five times, so Monday falls on July $1, 2,$ or $3.$ The days that occur five times in August are those of August $1, 2,$ $3,$ and August 1 is three weekdays after July $1.$

Testing the three cases, the August weekdays occurring five times are Thursday through Saturday, Wednesday through Friday, and Tuesday through Thursday. Thursday appears in every case.

Thus, the correct answer is **D**.

9. If a, b, c, d are positive real numbers such that a, b, c, d form an increasing arithmetic sequence and a, b, d form a geometric sequence, then $\frac{a}{d}$ is

A $\frac{1}{12}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

Let $b = a + r, c = a + 2r, d = a + 3r$. The geometric condition $b^2 = ad$ gives $(a + r)^2 = a(a + 3r)$, i.e. $r^2 = ar$, so $r = a$.

Then $d = a + 3a = 4a$ and $\frac{a}{d} = \frac{1}{4}$.

Thus, the correct answer is **C**.

10. How many different integers can be expressed as the sum of three distinct members of the set $\{1, 4, 7, 10, 13, 16, 19\}$?

A 13

B 16

C 24

D 30

E 35

Solution:

Every element is one more than a multiple of 3, so any sum of three of them is a multiple of 3. The smallest sum is $1 + 4 + 7 = 12$ and the largest is $13 + 16 + 19 = 48$, and every multiple of 3 between them is attainable.

There are 13 multiples of 3 from 12 to 48.

Thus, the correct answer is **A**.

11. The positive integers $A, B, A - B$, and $A + B$ are all prime numbers. The sum of these four primes is

A even

B divisible by 3

C divisible by 5

D divisible by 7

E prime

Solution:

$A - B$ and $A + B$ have the same parity; being prime, both are odd, so A and B have opposite parity. If A were even, then the prime A would equal 2, but the positive prime B would satisfy $B \geq 2$ and make $A - B \leq 0$. Hence A is odd and the even prime B is 2.

Then $A - 2, A, A + 2$ are three primes. One is divisible by 3, so that one must equal 3; the triple is $3, 5, 7$. Their sum together with 2 is $2 + 3 + 5 + 7 = 17$, a prime.

Thus, the correct answer is **E**.

12. For how many integers n is

$$\frac{n}{20-n}$$

the square of an integer?

A 1

B 2

C 3

D 4

E 10

Solution:

Set $\dfrac{n}{20-n} = k^2$. Solving, $n = \dfrac{20k^2}{k^2+1}$. Since k^2 and k^2+1 are coprime, k^2+1 must divide 20, which happens only for $|k| = 0, 1, 2, 3.$

The signs of k do not change n; these give $n = 0, 10, 16,$ and $18,$ which is four values.

Thus, the correct answer is **D**.

13. The sum of 18 consecutive positive integers is a perfect square. The smallest possible value of this sum is

- A 169
- B 225
- C 289
- D 361
- E 441

Solution:

The sum of $n, n+1, \ldots, n+17$ is $18n + \dfrac{17 \cdot 18}{2} = 9(2n+17)$. Since 9 is a perfect square, $2n+17$ must be one too.

The smallest positive integer n making $2n+17$ a perfect square is $n = 4$, giving $2n+17 = 25$ and a sum of $9 \cdot 25 = 225.$

Thus, the correct answer is **B**.

14. Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

A 8

B 9

C 10

D 12

E 16

Solution:

Each pair of circles meets in at most 2 points, and there are $\binom{4}{2} = 6$ pairs, giving at most $6 \cdot 2 = 12$ intersection points.

The bound is attainable by taking four circles in general position so that every pair crosses twice and no three pass through the same point.

Thus, the correct answer is **D**.

15. How many four-digit numbers N have the property that the three-digit number obtained by removing the leftmost digit is one ninth of N?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Let a be the leading digit and x the three-digit number after removing it, so $N = 1000a + x$. The condition $N = 9x$ gives $1000a = 8x$, i.e. $x = 125a$.

For $a = 1, \ldots, 7$ this makes x a three-digit number, while $a = 8$ gives $x = 1000$. So there are 7 such numbers.

Thus, the correct answer is **D**.

16. Juan rolls a fair regular octahedral die marked with the numbers 1 through 8. Then Amal rolls a fair six-sided die. What is the probability that the product of the two rolls is a multiple of 3?

A $\frac{1}{12}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{7}{12}$

E $\frac{2}{3}$

Solution:

The product is a multiple of 3 if and only if at least one die shows 3 or 6. The octahedral die avoids $3, 6$ with probability $\frac{6}{8} = \frac{3}{4}$, and the six-sided die avoids them with probability $\frac{4}{6} = \frac{2}{3}$.

So neither shows a multiple of 3 with probability $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$, and the answer is $1 - \frac{1}{2} = \frac{1}{2}$.

Thus, the correct answer is **C**.

17. Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one third as fast as Andy's mower. If they all start to mow their lawns at the same time, who will finish first?

- A Andy
- B Beth
- C Carlos
- D Andy and Carlos tie for first.
- E All three tie.

Solution:

Let Andy's lawn have area A; then Beth's is $\frac{A}{2}$ and Carlos' is $\frac{A}{3}$. With Carlos' rate R, Beth mows at $2R$ and Andy at $3R$.

Their times are $\frac{A}{3R}$, $\frac{\frac{A}{2}}{2R} = \frac{A}{4R}$, and $\frac{\frac{A}{3}}{R} = \frac{A}{3R}$ respectively. Beth's time $\frac{A}{4R}$ is the smallest, so Beth finishes first.

Thus, the correct answer is **B**.

18. A point P is randomly selected from the rectangular region with vertices $(0,0)$, $(2,0)$, $(2,1)$, $(0,1)$. What is the probability that P is closer to the origin than it is to the point $(3,1)$?

A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{4}{5}$

E 1

Solution:

The points closer to $(0,0)$ than to $(3,1)$ lie on the origin side of the perpendicular bisector of that segment, the line $3x+y=5$.

Within the rectangle, this region is a trapezoid whose parallel sides have lengths $\frac{5}{3}$ (at $y=0$) and $\frac{4}{3}$ (at $y=1$), so its area is $\frac{1}{2}\left(\frac{5}{3}+\frac{4}{3}\right)=\frac{3}{2}$. The rectangle has area 2, so the probability is $\frac{\frac{3}{2}}{2}=\frac{3}{4}$.

Thus, the correct answer is **C**.

19. If $a, b,$ and c are positive real numbers such that $a(b+c)=152, b(c+a)=162,$ and $c(a+b)=170,$ then abc is

- A 672
- B 688
- C 704
- D 720
- E 750

Solution:

Adding the three equations gives $2(ab+bc+ca)=484,$ so $ab+bc+ca=242.$ Subtracting each original equation from this yields $bc=90, ca=80,$ and $ab=72.$

Multiplying, $(abc)^2=90\cdot 80\cdot 72=720^2,$ and since $abc>0,$ we get $abc=720.$

Thus, the correct answer is **D**.

20. Let $\triangle XOY$ be a right-angled triangle with $m\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.

A 24

B 26

C 28

D 30

E 32

Solution:

Let $OM = a$ and $ON = b$. Since M, N are midpoints, $XN^2 = (2a)^2 + b^2 = 361$ and $YM^2 = a^2 + (2b)^2 = 484.$

Adding, $5(a^2 + b^2) = 845,$ so $a^2 + b^2 = 169.$ Then $MN = \sqrt{a^2 + b^2} = 13,$ and since MN is the midsegment joining M and $N,$ we have $XY = 2\,MN = 26.$

Thus, the correct answer is **B**.

21. For all positive integers n less than 2002, let

$$a_n = \begin{cases} 11, & 13 \mid n \text{ and } 14 \mid n; \\ 13, & 14 \mid n \text{ and } 11 \mid n; \\ 14, & 11 \mid n \text{ and } 13 \mid n; \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\sum_{n=1}^{2001} a_n$.

A 448

B 486

C 1560

D 2001

E 2002

Solution:

Since $2002 = 11 \cdot 13 \cdot 14$ with $11, 13, 14$ pairwise coprime, $a_n = 11$ when $182 \mid n$, $a_n = 13$ when $154 \mid n$, and $a_n = 14$ when $143 \mid n$ (and no $n < 2002$ is divisible by all three).

For $n \leq 2001$ there are 10 multiples of 182, 12 of 154, and 13 of 143. So the sum is

$$\begin{aligned} &11 \cdot 10 + 13 \cdot 12 + 14 \cdot 13 \\ &= 110 + 156 + 182 \\ &= 448. \end{aligned}$$

Thus, the correct answer is **A**.

22. For all integers n greater than 1, define

$$a_n = \frac{1}{\log_n 2002}.$$

Let $b = a_2 + a_3 + a_4 + a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. Then $b - c$ equals

A -2

B -1

C $\frac{1}{2002}$

D $\frac{1}{1001}$

E $\frac{1}{2}$

Solution:

By change of base, $a_n = \dfrac{1}{\log_n 2002} = \log_{2002} n$. So

$$b - c$$
$$= \log_{2002} \frac{2 \cdot 3 \cdot 4 \cdot 5}{10 \cdot 11 \cdot 12 \cdot 13 \cdot 14}.$$

The fraction equals $\dfrac{120}{240240} = \dfrac{1}{2002}$, so $b - c = \log_{2002} \dfrac{1}{2002} = -1.$

Thus, the correct answer is **B**.

23. In $\triangle ABC$, we have $AB = 1$ and $AC = 2$. Side BC and the median from A to BC have the same length. What is BC?

A $\frac{1+\sqrt{2}}{2}$

B $\frac{1+\sqrt{3}}{2}$

C $\sqrt{2}$

D $\frac{3}{2}$

E $\sqrt{3}$

Solution:

Let M be the midpoint of BC, set $AM = 2a$, and let $\theta = \angle AMB$, so $\angle AMC = 180^\circ - \theta$. With $BM = CM = a$, so that $BC = 2a$, the Law of Cosines in $\triangle ABM$ and $\triangle AMC$ gives

$$a^2 + 4a^2 - 4a^2 \cos\theta = 1,$$

$$a^2 + 4a^2 + 4a^2 \cos\theta = 4.$$

Adding, $10a^2 = 5$, so $a = \frac{\sqrt{2}}{2}$ and $BC = 2a = \sqrt{2}$.

Thus, the correct answer is **C**.

24. A convex quadrilateral $ABCD$ with area 2002 contains a point P in its interior such that $PA = 24$, $PB = 32$, $PC = 28$, and $PD = 45$. Find the perimeter of $ABCD$.

A $4\sqrt{2002}$

B $2\sqrt{8465}$

C $2\left(48+\sqrt{2002}\right)$

D $2\sqrt{8633}$

E $4\left(36+\sqrt{113}\right)$

Solution:

For any quadrilateral, the area is at most $\frac{1}{2}\, d_1 d_2$ where d_1, d_2 are the diagonals, with equality exactly when they are perpendicular. Here

$$\begin{aligned} 2002 &= \text{Area} \\ &\le \tfrac{1}{2}\, AC \cdot BD \\ &\le \tfrac{1}{2}(PA+PC)(PB+PD) \\ &= \tfrac{1}{2} \cdot 52 \cdot 77 \\ &= 2002. \end{aligned}$$

Equality forces the diagonals to be perpendicular and to intersect at P. Then

$$\begin{aligned} AB &= \sqrt{24^2+32^2} = 40, \\ BC &= \sqrt{28^2+32^2} = 4\sqrt{113}, \end{aligned}$$

$$\begin{aligned} CD &= \sqrt{28^2+45^2} = 53, \\ DA &= \sqrt{45^2+24^2} = 51. \end{aligned}$$

The perimeter is $144 + 4\sqrt{113} = 4\left(36+\sqrt{113}\right)$.

Thus, the correct answer is **E**.

25. Let $f(x) = x^2 + 6x + 1$, and let R denote the set of points (x, y) in the coordinate plane such that

$$f(x) + f(y) \leq 0$$

and

$$f(x) - f(y) \leq 0.$$

The area of R is closest to

A 21

B 22

C 23

D 24

E 25

Solution:

Completing the square,

$$f(x) + f(y)$$
$$= (x+3)^2 + (y+3)^2 - 16,$$

so the first condition is the disk of radius 4 centered at $(-3, -3)$.

Also

$$f(x) - f(y)$$
$$= (x-y)(x+y+6),$$

so the second condition $(x-y)(x+y+6) \leq 0$ describes two half-planes bounded by the perpendicular lines through $(-3, -3)$ of slopes 1 and -1. These cut the disk into two equal halves.

Thus R has area $\frac{1}{2}\pi \cdot 4^2 = 8\pi \approx 25.13$, which is closest to 25.

Thus, the correct answer is **E**.

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