

2004 AMC 12B Solutions

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1. At each basketball practice last week, Jenny made twice as many free throws as she made at the previous practice. At her fifth practice she made 48 free throws. How many free throws did she make at the first practice?

☒ A 3

☐ B 6

☐ C 9

☐ D 12

☐ E 15

Solution:

Each practice she made twice the previous, so going backward we halve. From the fifth practice at 48, the earlier practices had 24, 12, 6, and 3 free throws.

Thus, the correct answer is **A**.

2. In the expression $c \cdot a^b - d$, the values of a , b , c , and d are 0, 1, 2, and 3, although not necessarily in that order. What is the maximum possible value of the result?

- ☐ A 5
- ☐ B 6
- ☐ C 8
- ☒ D 9
- ☐ E 10

Solution:

To maximize, set $d = 0$. With a , b , c taking 1, 2, 3, the term $c \cdot a^b$ is largest when $c = 1$ and $a^b = 3^2 = 9$. This gives $1 \cdot 9 - 0 = 9$, which beats $2^3 = 8$ and the other assignments.

Thus, the correct answer is **D**.

3. If x and y are positive integers for which $2^x 3^y = 1296$, what is the value of $x + y$?

A 8

B 9

C 10

D 11

E 12

Solution:

Factoring, $1296 = 6^4 = 2^4 \cdot 3^4$. Matching exponents gives $x = 4$ and $y = 4$, so $x + y = 8$.

Thus, the correct answer is **A**.

4. An integer x , with $10 \leq x \leq 99$, is to be chosen. If all choices are equally likely, what is the probability that at least one digit of x is a 7?

A $\frac{1}{9}$

B $\frac{1}{5}$

C $\frac{19}{90}$

D $\frac{2}{9}$

E $\frac{1}{3}$

Solution:

There are 90 integers from 10 to 99. Ten have a units digit 7, and nine have a tens digit 7. Since 77 is counted twice, there are $10 + 9 - 1 = 18$ with at least one 7. The probability is $\frac{18}{90} = \frac{1}{5}$.

Thus, the correct answer is **B**.

5. On a trip from the United States to Canada, Isabella took d U.S. dollars. At the border she exchanged them all, receiving 10 Canadian dollars for every 7 U.S. dollars. After spending 60 Canadian dollars, she had d Canadian dollars left. What is the sum of the digits of d ?

A 5

B 6

C 7

D 8

E 9

Solution:

Exchanging gives $\frac{10d}{7}$ Canadian dollars. After spending 60, she has $\frac{10d}{7} - 60 = d$.

Then $\frac{3d}{7} = 60$, so $d = 140$. The sum of its digits is $1 + 4 + 0 = 5$.

Thus, the correct answer is **A**.

6. Minneapolis–St. Paul International Airport is 8 miles southwest of downtown St. Paul and 10 miles southeast of downtown Minneapolis. Which of the following is closest to the number of miles between downtown St. Paul and downtown Minneapolis?

A 13

B 14

C 15

D 16

E 17

Solution:

Southwest and southeast are perpendicular, so the airport sits at the right angle of a right triangle with legs 8 and 10. The distance between downtowns is $\sqrt{10^2 + 8^2} = \sqrt{164} \approx 12.8$, closest to 13.

Thus, the correct answer is **A**.

7. A square has sides of length 10, and a circle centered at one of its vertices has radius 10. What is the area of the union of the regions enclosed by the square and the circle?

A $200 + 25\pi$

B $100 + 75\pi$

C $75 + 100\pi$

D $100 + 100\pi$

E $100 + 125\pi$

Solution:

The square has area 100 and the circle has area 100π . Their overlap is the quarter of the circle lying inside the square, with area 25π . The union is $100 + 100\pi - 25\pi = 100 + 75\pi$.

Thus, the correct answer is **B**.

8. A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

- A 5
- B 8
- C 9
- D 10**
- E 11

Solution:

The rows contain $1, 3, 5, \dots, (2n - 1)$ cans, and the sum of the first n odd numbers is n^2 . Setting $n^2 = 100$ gives $n = 10$.

Thus, the correct answer is **D**.

9. The point $(-3, 2)$ is rotated 90° clockwise around the origin to point B . Point B is then reflected in the line $y = x$ to point C . What are the coordinates of C ?

A $(-3, -2)$

B $(-2, -3)$

C $(2, -3)$

D $(2, 3)$

E $(3, 2)$

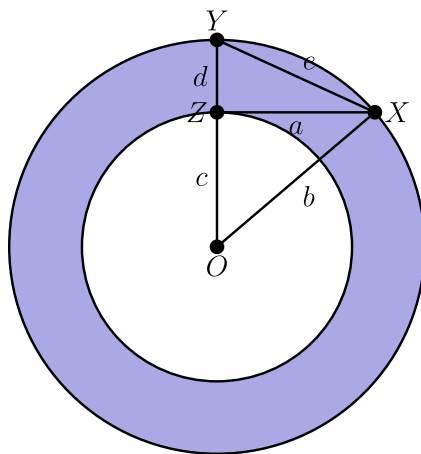
Solution:

Rotating $(-3, 2)$ by 90° clockwise sends $(x, y) \rightarrow (y, -x)$, giving $B = (2, 3)$.

Reflecting in $y = x$ swaps coordinates, giving $C = (3, 2)$.

Thus, the correct answer is **E**.

10. An *annulus* is the region between two concentric circles. The concentric circles in the figure have radii b and c , with $b > c$. Let $\overline{OX}$ be a radius of the larger circle, let $\overline{XZ}$ be tangent to the smaller circle at Z , and let $\overline{OY}$ be the radius of the larger circle that contains Z . Let $a = XZ$, $d = YZ$, and $e = XY$. What is the area of the annulus?



A πa^2

B πb^2

C πc^2

D πd^2

E πe^2

Solution:

The annulus area is $\pi b^2 - \pi c^2$. Because $\overline{XZ}$ is tangent to the smaller circle at Z , it is perpendicular to radius $\overline{OZ}$, so $\triangle OZX$ is right-angled at Z . Then $b^2 = c^2 + a^2$, giving $b^2 - c^2 = a^2$. The area is πa^2 .

Thus, the correct answer is **A**.

11. All the students in an algebra class took a 100-point test. Five students scored 100, each student scored at least 60, and the mean score was 76. What is the smallest possible number of students in the class?

A 10

B 11

C 12

D 13

E 14

Solution:

Each score of 100 is 24 above the mean, so the five contribute 120 points above 76. These must be balanced by points below the mean, and each remaining student is at most $76 - 60 = 16$ below. So at least $\frac{120}{16} = 7.5$, hence 8 more students are needed, for a total of 13. Five 100s and eight 61s achieve this.

Thus, the correct answer is **D**.

12. In the sequence 2001, 2002, 2003, $\dots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004th term in this sequence?

A -2004

B -2

C 0

D 4003

E 6007

Solution:

The rule gives 2001, 2002, 2003, 2000, 2005, 1998, $\dots$. The even-indexed terms are 2002, 2000, 1998, $\dots$, decreasing by 2.

More precisely, the recurrence verifies inductively that $a_{2k} = 2004 - 2k$ and $a_{2k+1} = 2001 + 2k$. Therefore

$$\begin{aligned} a_{2004} &= a_{2 \cdot 1002} \\ &= 2004 - 2(1002) = 0. \end{aligned}$$

Thus, the correct answer is **C**.

13. If $f(x) = ax + b$ and $f^{-1}(x) = bx + a$ with a and b real, what is the value of $a + b$?

A -2

B -1

C 0

D 1

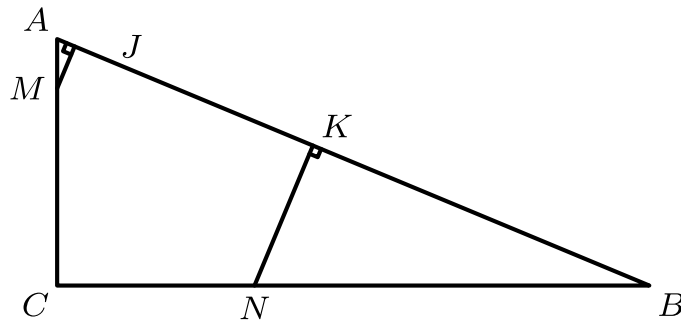
E 2

Solution:

Since $f(f^{-1}(x)) = x$, we have $a(bx + a) + b = x$. Matching terms gives $ab = 1$ and $a^2 + b = 0$. Then $b = \frac{1}{a}$ and $a^2 + \frac{1}{a} = 0$, so $a^3 = -1$, giving $a = -1$ and $b = -1$. Thus $a + b = -2$.

Thus, the correct answer is **A**.

14. In $\triangle ABC$, $AB = 13$, $AC = 5$ and $BC = 12$. Points M and N lie on $\overline{AC}$ and $\overline{BC}$, respectively, with $CM = CN = 4$. Points J and K are on $\overline{AB}$ so that $\overline{MJ}$ and $\overline{NK}$ are perpendicular to $\overline{AB}$. What is the area of pentagon $CMJKN$?



- ☐ A 15
- ☐ B $\frac{81}{5}$
- ☐ C $\frac{205}{12}$
- ☒ D $\frac{240}{13}$
- ☐ E 20

Solution:

Since $5^2 + 12^2 = 13^2$, $\triangle ABC$ is right-angled at C with area $\frac{1}{2}(5)(12) = 30$. The small right triangles $\triangle AMJ$ and $\triangle NBK$ are each similar to $\triangle ABC$, with hypotenuses $AM = 5 - 4 = 1$ and $BN = 12 - 4 = 8$. Their areas are $\left(\frac{1}{13}\right)^2 (30)$ and $\left(\frac{8}{13}\right)^2 (30)$.

The pentagon is what remains:

$$\begin{aligned} & \left(1 - \frac{1}{169} - \frac{64}{169}\right) (30) \\ &= \frac{104}{169} (30) = \frac{240}{13}. \end{aligned}$$

Thus, the correct answer is **D**.

- 15.** The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years Jack will be twice as old as Bill will be then. What is the difference in their current ages?

- ☐ A 9
- ☒ B 18
- ☐ C 27
- ☐ D 36
- ☐ E 45

Solution:

Let Jack be $10x + y$ and Bill be $10y + x$. Then $10x + y + 5 = 2(10y + x + 5)$, so $8x = 19y + 5$. Testing digits, only $y = 1, x = 3$ works, so Jack is **31** and Bill is **13**. The difference is $31 - 13 = 18$.

Thus, the correct answer is **B**.

16. A function f is defined by $f(z) = i\bar{z}$, where $i = \sqrt{-1}$ and $\bar{z}$ is the complex conjugate of z . How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

A 0

B 1

C 2

D 4

E 8

Solution:

Writing $z = x + iy$, we get $f(z) = i(x - iy) = y + ix$. Setting $f(z) = z$ gives $y = x$, which is a line through the origin. The condition $|z| = 5$ is a circle, and a line through the center meets the circle in 2 points.

Thus, the correct answer is **C**.

17. For some real numbers a and b , the equation

$$8x^3 + 4ax^2 + 2bx + a = 0$$

has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of a ?

A -256

B -64

C -8

D 64

E 256

Solution:

The sum of the base-2 logarithms is $\log_2(r_1 r_2 r_3) = 5$, so $r_1 r_2 r_3 = 2^5 = 32$. By Vieta's formulas on $8x^3 + 4ax^2 + 2bx + a$, the product of the roots is $-\frac{a}{8}$. Thus $-\frac{a}{8} = 32$, giving $a = -256$.

Thus, the correct answer is **A**.

18. Points A and B are on the parabola $y = 4x^2 + 7x - 1$, and the origin is the midpoint of $\overline{AB}$. What is the length of AB ?

A $2\sqrt{5}$

B $5 + \frac{\sqrt{2}}{2}$

C $5 + \sqrt{2}$

D 7

E $5\sqrt{2}$

Solution:

Let $B = (a, b)$ and $A = (-a, -b)$. Then $4a^2 + 7a - 1 = b$ and $4a^2 - 7a - 1 = -b$.

Subtracting gives $14a = 2b$, so $b = 7a$. Then $4a^2 + 7a - 1 = 7a$ gives $a^2 = \frac{1}{4}$, and

$$b^2 = 49a^2 = \frac{49}{4}. \text{ So } AB = 2\sqrt{a^2 + b^2} = 2\sqrt{\frac{50}{4}} = 5\sqrt{2}.$$

Thus, the correct answer is **E**.

19. A truncated cone has horizontal bases with radii 18 and 2. A sphere is tangent to the top, bottom, and lateral surface of the truncated cone. What is the radius of the sphere?

A 6

B $4\sqrt{5}$

C 9

D 10

E $6\sqrt{3}$

Solution:

The axial cross-section is a trapezoid $ABCD$ with parallel sides 2 and 18 and an inscribed circle (a great circle of the sphere). By equal tangent lengths from B and C , the slant side $BC = 18 + 2 = 20$. Dropping a perpendicular from C to the bottom base gives a right triangle with horizontal leg $18 - 2 = 16$, so the height is $\sqrt{20^2 - 16^2} = 12$. The sphere's radius is half the height, 6.

Thus, the correct answer is **A**.

20. Each face of a cube is painted either red or blue, each with probability $\frac{1}{2}$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

A $\frac{1}{4}$

B $\frac{5}{16}$

C $\frac{3}{8}$

D $\frac{7}{16}$

E $\frac{1}{2}$

Solution:

There are $2^6 = 64$ colorings. A suitable orientation exists when all six faces are one color (2 ways), exactly five faces are one color ($2 \cdot 6 = 12$ ways), or four faces are one color with the other color on a pair of opposite faces ($2 \cdot 3 = 6$ ways). That is $2 + 12 + 6 = 20$ favorable colorings, so the probability is $\frac{20}{64} = \frac{5}{16}$.

Thus, the correct answer is **B**.

21. The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the xy -plane. Let a and b be the maximum and minimum values of $\frac{y}{x}$ over all points (x, y) on the ellipse. What is the value of $a + b$?

A 3

B $\sqrt{10}$

C $\frac{7}{2}$

D $\frac{9}{2}$

E $2\sqrt{14}$

Solution:

The slopes a and b are the values of m for which $y = mx$ meets the ellipse in exactly one point. Substituting gives

$$(3m^2 + m + 2)x^2 - (20m + 11)x + 40 = 0.$$

Setting its discriminant to zero yields $-80m^2 + 280m - 199 = 0$. By Vieta's formulas, $a + b = \frac{280}{80} = \frac{7}{2}$.

Thus, the correct answer is **C**.

22. The square

50	b	c
d	e	f
g	h	2

is a multiplicative magic square. That is, the product of the numbers in each row, column, and diagonal is the same. If all the entries are positive integers, what is the sum of the possible values of g ?

- A 10
- B 25
- C 35**
- D 62
- E 136

Solution:

From the equal row, column, and diagonal products, every entry can be written in terms of b : $h = \frac{100}{b}$, $g = \frac{100}{c}$, $f = \frac{100}{d}$. Comparing rows and columns gives $c = \frac{20}{b}$ and $d = \frac{4}{b}$, hence $g = 5b$ and $e = 10$.

All entries are positive integers exactly when $b = 1, 2$, or 4 , giving $g = 5, 10, 20$. Their sum is **35**.

Thus, the correct answer is **C**.

23. The polynomial $x^3 - 2004x^2 + mx + n$ has integer coefficients and three distinct positive zeros. Exactly one of these is an integer, and it is the sum of the other two. How many values of n are possible?

A 250,000

B 250,250

C 250,500

D 250,750

E 251,000

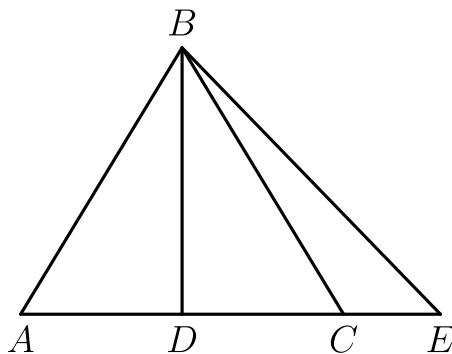
Solution:

Let the integer zero be a . The other two zeros are irrational conjugates $\frac{a}{2} \pm r$, whose sum a equals the integer zero. Vieta's formula on the x^2 coefficient gives $a + a = 2004$, so $a = 1002$ and the conjugate pair is $501 \pm r$.

The coefficients are integers exactly when r^2 is a positive integer, and the zeros are positive and distinct when $1 \leq r^2 \leq 501^2 - 1 = 251,000$. Since r cannot be an integer, we exclude the 500 perfect-square values $r^2 = 1^2, \dots, 500^2$, leaving $251,000 - 500 = 250,500$ values of n .

Thus, the correct answer is **C**.

24. In $\triangle ABC$, $AB = BC$, and $\overline{BD}$ is an altitude. Point E is on the extension of $\overline{AC}$ such that $BE = 10$. The values of $\tan \angle CBE$, $\tan \angle DBE$, and $\tan \angle ABE$ form a geometric progression, and the values of $\cot \angle DBE$, $\cot \angle CBE$, $\cot \angle DBC$ form an arithmetic progression. What is the area of $\triangle ABC$?



- A 16
- B $\frac{50}{3}$**
- C $10\sqrt{3}$
- D $8\sqrt{5}$
- E 18

Solution:

Let $\angle DBE = \alpha$ and $\angle DBC = \beta$. Since $\overline{BD}$ is the altitude of the isosceles triangle, $\angle CBE = \alpha - \beta$ and $\angle ABE = \alpha + \beta$. The geometric progression gives $\tan(\alpha - \beta) \tan(\alpha + \beta) = \tan^2 \alpha$, which simplifies to $\tan^2 \beta (\tan^4 \alpha - 1) = 0$, so $\tan \alpha = 1$ and $\alpha = 45^\circ$.

Writing $DC = a$ and $BD = b$, the arithmetic progression $\cot \angle DBE$, $\cot \angle CBE$, $\cot \angle DBC$ becomes $1, \frac{b+a}{b-a}, \frac{b}{a}$, forcing $b = 3a$. With $BE = 10$ and $\angle DBE = 45^\circ$, we get $b = \frac{BE}{\sqrt{2}} = 5\sqrt{2}$, so $a = \frac{5\sqrt{2}}{3}$.

The area of $\triangle ABC$ is $\frac{1}{2}(AC)(BD) = ab = 5\sqrt{2} \cdot \frac{5\sqrt{2}}{3} = \frac{50}{3}$.

Thus, the correct answer is **B**.

25. Given that 2^{2004} is a 604-digit number whose first digit is 1, how many elements of the set $S = \{2^0, 2^1, 2^2, \dots, 2^{2003}\}$ have a first digit of 4?

A 194

B 195

C 196

D 197

E 198

Solution:

The smallest power of 2 with any given digit-count lies between 10^k and $2 \cdot 10^k$, so it has leading digit 1. Because 2^{2004} is a 604-digit number beginning with 1, it is the first 604-digit power. Thus the powers in S contain exactly one leading-1 number for each digit-count from 1 through 603, or 603 in all.

After each leading-1 power, the next power leads with 2 or 3, and the following power leads with 4, 5, 6, or 7. Hence 603 elements lead with 2 or 3, another 603 lead with 4 through 7, and the remaining $2004 - 3(603) = 195$ lead with 8 or 9.

Finally, halving a power that leads with 8 or 9 produces the preceding power with leading digit 4, and doubling any leading-4 power reverses this. This is a bijection, so there are 195 elements with first digit 4.

Thus, the correct answer is **B**.

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