

2006 AMC 12B Solutions

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1. What is

$$(-1)^1 + (-1)^2 + \cdots + (-1)^{2006}?$$

A -2006

B -1

C 0

D 1

E 2006

Solution:

Since $(-1)^k = -1$ for odd k and $(-1)^k = 1$ for even k , the terms alternate $-1, 1, -1, 1, \dots$

There are **2006** terms, forming **1003** pairs, each equal to $(-1) + 1 = 0$. The total is **0**.

Thus, the correct answer is **C**.

2. For real numbers x and y , define

$$x \spadesuit y = (x + y)(x - y).$$

What is $3 \spadesuit (4 \spadesuit 5)$?

☒ A -72

☐ B -27

☐ C -24

☐ D 24

☐ E 72

Solution:

Since $x \spadesuit y = x^2 - y^2$, the inner value is $4 \spadesuit 5 = 16 - 25 = -9$.

Then $3 \spadesuit (-9) = 3^2 - (-9)^2 = 9 - 81 = -72$.

Thus, the correct answer is **A**.

3. A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of 34 points, and the Cougars won by a margin of 14 points. How many points did the Panthers score?

A 10

B 14

C 17

D 20

E 24

Solution:

Let c and p be the Cougars' and Panthers' scores. Then $c + p = 34$ and $c - p = 14$.

Subtracting gives $2p = 20$, so $p = 10$.

Thus, the correct answer is **A**.

4. Mary is about to pay for five items at the grocery store. The prices of the items are \$7.99, \$4.99, \$2.99, \$1.99, and \$0.99. Mary will pay with a twenty-dollar bill. Which of the following is closest to the percentage of the \$20.00 that she will receive in change?

A 5

B 10

C 15

D 20

E 25

Solution:

The prices total \$18.95, so the change is \$1.05. As a percentage of \$20.00, this is

$$\frac{1.05}{20} \cdot 100\% = 5.25\%.$$

The closest listed percentage is 5%.

Thus, the correct answer is **A**.

5. John is walking east at a speed of 3 miles per hour, while Bob is also walking east, but at a speed of 5 miles per hour. If Bob is now 1 mile west of John, how many minutes will it take for Bob to catch up to John?

A 30

B 50

C 60

D 90

E 120

Solution:

Bob closes the gap at a relative speed of $5 - 3 = 2$ miles per hour. To cover the 1-mile gap takes

$$\frac{1}{2}$$

hour, or 30 minutes.

Thus, the correct answer is **A**.

6. Francesca uses 100 grams of lemon juice, 100 grams of sugar, and 400 grams of water to make lemonade. There are 25 calories in 100 grams of lemon juice and 386 calories in 100 grams of sugar. Water contains no calories. How many calories are in 200 grams of her lemonade?

A 129

B 137

C 174

D 223

E 411

Solution:

The full batch weighs $100 + 100 + 400 = 600$ grams and contains $25 + 386 = 411$ calories.

Since 200 grams is one third of the batch, it has

$$\frac{411}{3} = 137$$

calories.

Thus, the correct answer is **B**.

7. Mr. and Mrs. Lopez have two children. When they get into their family car, two people sit in the front, and the other two sit in the back. Either Mr. Lopez or Mrs. Lopez must sit in the driver's seat. How many seating arrangements are possible?

- ☐ A 4
- ☒ B 12
- ☐ C 16
- ☐ D 24
- ☐ E 48

Solution:

The driver is one of the two parents: **2** choices.

Any of the remaining **3** people can sit in the front passenger seat, and the last **2** people fill the back in **2** orders.

The total is $2 \cdot 3 \cdot 2 = 12$.

Thus, the correct answer is **B**.

8. The lines

$$x = \frac{1}{4}y + a, \quad y = \frac{1}{4}x + b$$

intersect at the point $(1, 2)$. What is $a + b$?

☐ A 0

☐ B $\frac{3}{4}$

☐ C 1

☐ D 2

☒ E $\frac{9}{4}$

Solution:

Substituting $(1, 2)$ gives

$$1 = \frac{2}{4} + a \Rightarrow a = \frac{1}{2},$$

and

$$2 = \frac{1}{4} + b \Rightarrow b = \frac{7}{4}.$$

Therefore

$$a + b = \frac{1}{2} + \frac{7}{4} = \frac{9}{4}.$$

Thus, the correct answer is **E**.

9. How many even three-digit integers have the property that their digits, read left to right, are in strictly increasing order?

A 21

B 34

C 51

D 72

E 150

Solution:

Let the digits be $a < b < c$ with c even. Since $a \geq 1$, no digit is zero, and $c \neq 2$ (there is no room for two smaller nonzero digits).

Once the units digit c is fixed, any two distinct digits below it can be arranged in increasing order in exactly one way. So the count for each c is $\binom{c-1}{2}$.

For $c = 4, 6, 8$ this gives

$$\begin{aligned} \binom{3}{2} + \binom{5}{2} \\ + \binom{7}{2} &= 3 + 10 + 21 \\ &= 34. \end{aligned}$$

Thus, the correct answer is **B**.

10. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

A 43

B 44

C 45

D 46

E 47

Solution:

Let the sides be x , $3x$, and 15. The triangle inequality requires $x + 3x > 15$, so $x \geq 4$, and $x + 15 > 3x$, so $x \leq 7$.

The perimeter $4x + 15$ is largest when $x = 7$, giving $7 + 21 + 15 = 43$.

Thus, the correct answer is **A**.

11. Joe and JoAnn each bought 12 ounces of coffee in a 16-ounce cup. Joe drank 2 ounces of his coffee and then added 2 ounces of cream. JoAnn added 2 ounces of cream, stirred the coffee well, and then drank 2 ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

A $\frac{6}{7}$

B $\frac{13}{14}$

C 1

D $\frac{14}{13}$

E $\frac{7}{6}$

Solution:

Joe adds the cream last, so his cup holds all 2 ounces of cream.

JoAnn's cup has 14 ounces of mixture containing 2 ounces of cream. Drinking 2 ounces removes a fraction $\frac{2}{14}$ of everything, leaving

$$2 \cdot \frac{12}{14} = \frac{12}{7}$$

ounces of cream.

The ratio is

$$\frac{\frac{2}{12}}{\frac{12}{7}} = \frac{14}{12} = \frac{7}{6}.$$

Thus, the correct answer is **E**.

12. The parabola $y = ax^2 + bx + c$ has vertex (p, p) and y -intercept $(0, -p)$, where $p \neq 0$. What is b ?

A $-p$

B 0

C 2

D 4

E p

Solution:

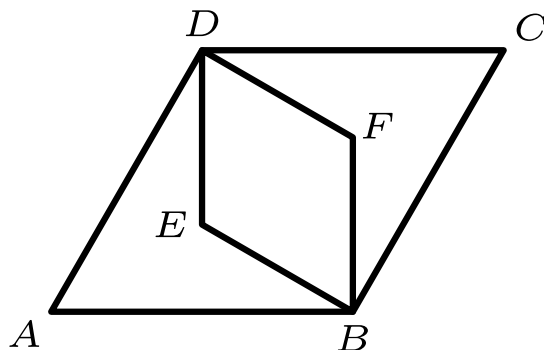
The vertex form is $y = a(x - p)^2 + p$.

At $x = 0$, $y = ap^2 + p = -p$, so $ap^2 = -2p$ and $a = -\frac{2}{p}$.

Expanding, $y = ax^2 - 2apx + ap^2 + p$, so $b = -2ap = -2\left(-\frac{2}{p}\right)p = 4$.

Thus, the correct answer is **D**.

13. Rhombus $ABCD$ is similar to rhombus $BFDE$. The area of rhombus $ABCD$ is 24, and $\angle BAD = 60^\circ$. What is the area of rhombus $BFDE$?



- ☐ A 6
- ☐ B $4\sqrt{3}$
- ☒ C 8
- ☐ D 9
- ☐ E $6\sqrt{3}$

Solution:

Let the diagonals of $ABCD$ meet at O . They bisect each other at right angles, and since $\angle BAD = 60^\circ$, triangle ABO is a 30° - 60° - 90° triangle. Hence the half-diagonals satisfy $AO = \sqrt{3} BO$.

The segment BD is the short diagonal of $ABCD$ and the long diagonal of the similar rhombus $BFDE$. Thus the smaller-to-larger length ratio is $\frac{BD}{AC} = \frac{BO}{AO} = \frac{1}{\sqrt{3}}$. Areas scale by the square of this ratio, so the area of $BFDE$ is $24 \cdot \frac{1}{3} = 8$.

Thus, the correct answer is **C**.

14. Elmo makes N sandwiches for a fundraiser. For each sandwich he uses B globs of peanut butter at 4¢ per glob and J blobs of jam at 5¢ per blob. The cost of the peanut butter and jam to make all the sandwiches is \$2.53. Assume that B , J , and N are positive integers with $N > 1$. What is the cost of the jam Elmo uses to make the sandwiches?

A \$1.05

B \$1.25

C \$1.45

D \$1.65

E \$1.85

Solution:

The total cost in cents is $N(4B + 5J) = 253 = 11 \cdot 23$. Since $N > 1$, the value of N is 11, 23, or 253.

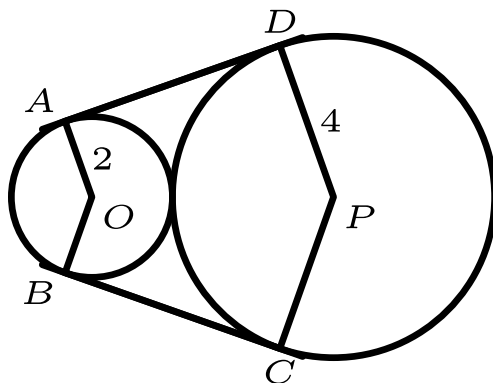
If $N = 253$ then $4B + 5J = 1$, and if $N = 23$ then $4B + 5J = 11$; neither has a positive integer solution.

So $N = 11$ and $4B + 5J = 23$, whose only positive solution is $B = 2$, $J = 3$.

The jam costs $N \cdot J \cdot 5\text{¢} = 11 \cdot 3 \cdot 5 = 165$ cents, or \$1.65.

Thus, the correct answer is **D**.

15. Circles with centers O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B are on the circle centered at O , and points C and D are on the circle centered at P , such that AD and BC are common external tangents to the circles. What is the area of hexagon $AOBCPD$?



- A $18\sqrt{3}$
- B $24\sqrt{2}$**
- C 36
- D $24\sqrt{3}$
- E $32\sqrt{2}$

Solution:

The circles are externally tangent, so $OP = 2 + 4 = 6$. In quadrilateral $AOPD$, both $OA = 2$ and $PD = 4$ are perpendicular to the tangent line AD , making it a right trapezoid.

Drawing the line through O parallel to AD creates a right triangle with hypotenuse $OP = 6$ and one leg $PD - OA = 2$, so $AD = \sqrt{6^2 - 2^2} = \sqrt{32} = 4\sqrt{2}$.

The trapezoid $AOPD$ has area

$$\frac{1}{2}(2 + 4)(4\sqrt{2}) = 12\sqrt{2}.$$

By symmetry the hexagon $AOBCPD$ is made of two such trapezoids, so its area is $2 \cdot 12\sqrt{2} = 24\sqrt{2}$.

Thus, the correct answer is **B**.

16. Regular hexagon $ABCDEF$ has vertices A and C at $(0, 0)$ and $(7, 1)$, respectively. What is its area?

A $20\sqrt{3}$

B $22\sqrt{3}$

C $25\sqrt{3}$

D $27\sqrt{3}$

E 50

Solution:

The distance is $AC = \sqrt{7^2 + 1^2} = \sqrt{50}$. In a regular hexagon with side s , the distance between vertices two apart is $s\sqrt{3}$, so $s^2 \cdot 3 = 50$, giving $s^2 = \frac{50}{3}$.

The hexagon's area is

$$\frac{3\sqrt{3}}{2}s^2 = \frac{3\sqrt{3}}{2} \cdot \frac{50}{3} = 25\sqrt{3}.$$

Thus, the correct answer is **C**.

17. For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5, and 6 on each die are in the ratio 1 : 2 : 3 : 4 : 5 : 6. What is the probability of rolling a total of 7 on the two dice?

A $\frac{4}{63}$

B $\frac{1}{8}$

C $\frac{8}{63}$

D $\frac{1}{6}$

E $\frac{2}{7}$

Solution:

Since the weights sum to 21, the probability of rolling k is $\frac{k}{21}$.

A total of 7 comes from (1, 6), (2, 5), ..., (6, 1), so the probability is

$$\frac{1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1}{21^2} \\ = \frac{56}{441} = \frac{8}{63}.$$

Thus, the correct answer is **C**.

18. An object in the plane moves from one lattice point to another. At each step, the object may move one unit to the right, one unit to the left, one unit up, or one unit down. If the object starts at the origin and takes a ten-step path, how many different points could be the final point?

A 120

B 121

C 221

D 230

E 231

Solution:

Each step changes the coordinate sum by 1, so after 10 steps the endpoint (a, b) has $a + b$ even, and $|a| + |b| \leq 10$. Any such point is reachable: walk $|a| + |b|$ steps to it, then use the remaining even number of steps going out and back.

The reachable points lie on the lines $a + b = 2k$ for $-5 \leq k \leq 5$. Each such line meets the diamond in exactly 11 lattice points.

With 11 lines and 11 points each, there are 121 points.

Thus, the correct answer is **B**.

19. Mr. Jones has eight children of different ages. On a family trip his oldest child, who is 9, spots a license plate with a 4-digit number in which each of two digits appears two times. "Look, daddy!" she exclaims. "That number is evenly divisible by the age of each of us kids!" "That's right," replies Mr. Jones, "and the last two digits just happen to be my age." Which of the following is *not* the age of one of Mr. Jones's children?

A 4

B 5

C 6

D 7

E 8

Solution:

The number has the form $aabb$, $abab$, or $baab$. Divisibility by 9 means $2(a + b)$ is a multiple of 9, so $a + b = 9$.

The eight distinct ages are eight of the nine integers from 1 through 9, so at least one of ages 4 and 8 occurs. Therefore the number is divisible by 4. The possibilities become 1188, 2772, 3636, 5544, 6336, 7272, 9900.

Since the last two digits are Mr. Jones's age, 9900 is impossible, and none of the others is a multiple of 5. So the children's ages cannot include 5. Indeed 5544 is divisible by 1, 2, 3, 4, 6, 7, 8, 9.

Thus, the correct answer is **B**.

20. Let x be chosen at random from the interval $(0, 1)$. What is the probability that

$$\lfloor \log_{10} 4x \rfloor - \lfloor \log_{10} x \rfloor = 0?$$

Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x .

A $\frac{1}{8}$

B $\frac{3}{20}$

C $\frac{1}{6}$

D $\frac{1}{5}$

E $\frac{1}{4}$

Solution:

The equation says $\lfloor \log_{10} x \rfloor = \lfloor \log_{10} 4x \rfloor$, i.e. x and $4x$ lie in the same interval $[10^n, 10^{n+1})$.

This holds exactly when $10^n \leq x$ and $4x < 10^{n+1}$, that is $10^n \leq x < \frac{10^{n+1}}{4}$.

Within $[10^n, 10^{n+1})$, the favorable fraction is

$$\begin{aligned} & \frac{\frac{10^{n+1}}{4} - 10^n}{10^{n+1} - 10^n} \\ &= \frac{\frac{10}{4} - 1}{10 - 1} = \frac{1}{6}. \end{aligned}$$

Since this fraction is the same on every such interval, the overall probability is $\frac{1}{6}$.

Thus, the correct answer is **C**.

21. Rectangle $ABCD$ has area 2006. An ellipse with area 2006π passes through A and C and has foci at B and D . What is the perimeter of the rectangle? (The area of an ellipse is πab , where $2a$ and $2b$ are the lengths of its axes.)

A $\frac{16\sqrt{2006}}{\pi}$

B $\frac{1003}{4}$

C $8\sqrt{1003}$

D $6\sqrt{2006}$

E $\frac{32\sqrt{1003}}{\pi}$

Solution:

Let the rectangle's sides be x and y . Point A is on the ellipse with foci B and D , so $x + y = AB + AD = 2a$. The distance between the foci is the diagonal, so $\sqrt{x^2 + y^2} = 2\sqrt{a^2 - b^2}$.

Then $2xy = (x + y)^2 - (x^2 + y^2) = 4a^2 - (4a^2 - 4b^2) = 4b^2$, so $xy = 2b^2$. The area gives $2b^2 = 2006$, hence $b^2 = 1003$.

The ellipse area gives $\pi ab = 2006\pi$, so $ab = 2006$ and $a = \frac{2006}{\sqrt{1003}} = 2\sqrt{1003}$.

The perimeter is $2(x + y) = 4a = 8\sqrt{1003}$.

Thus, the correct answer is **C**.

22. Suppose a , b , and c are positive integers with $a + b + c = 2006$, and $a! b! c! = m \cdot 10^n$, where m and n are integers and m is not divisible by 10. What is the smallest possible value of n ?

A 489

B 492

C 495

D 498

E 501

Solution:

Since factors of 2 are more plentiful than factors of 5, n equals the number of factors of 5 in $a! b! c!$, namely

$$n = \sum_{k \geq 1} \left(\left\lfloor \frac{a}{5^k} \right\rfloor + \left\lfloor \frac{b}{5^k} \right\rfloor + \left\lfloor \frac{c}{5^k} \right\rfloor \right).$$

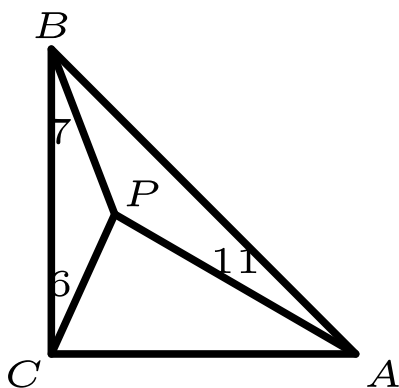
For each k , $\left\lfloor \frac{a}{5^k} \right\rfloor + \left\lfloor \frac{b}{5^k} \right\rfloor + \left\lfloor \frac{c}{5^k} \right\rfloor \geq \left\lfloor \frac{2006}{5^k} \right\rfloor - 2$. Summing over $k = 1, 2, 3, 4$ (as $2006 < 5^5$) gives

$$\begin{aligned} n &\geq (401 + 80 + 16 + 3) \\ &\quad - 4 \cdot 2 = 492. \end{aligned}$$

Equality is attainable, for example with $a = b = 624$ and $c = 758$. So the minimum is 492.

Thus, the correct answer is **B**.

23. Isosceles $\triangle ABC$ has a right angle at C . Point P is inside $\triangle ABC$, such that $PA = 11$, $PB = 7$, and $PC = 6$. Legs $\overline{AC}$ and $\overline{BC}$ have length $s = \sqrt{a + b\sqrt{2}}$, where a and b are positive integers. What is $a + b$?



- ☐ A 85
- ☐ B 91
- ☐ C 108
- ☐ D 121
- ☒ E 127

Solution:

Rotate $\triangle ABC$ by 90° about C , sending A to B and P to P' . Then $CP' = CP = 6$ and $\angle PCP' = 90^\circ$, so $\triangle PCP'$ is an isosceles right triangle with $PP' = 6\sqrt{2}$.

Also $BP' = AP = 11$. Since $(6\sqrt{2})^2 + 7^2 = 72 + 49 = 121 = 11^2$, triangle BPP' has a right angle at P . Hence $\angle BPC = \angle BPP' + \angle P'PC = 90^\circ + 45^\circ = 135^\circ$.

By the Law of Cosines in $\triangle BPC$,

$$\begin{aligned} BC^2 &= 6^2 + 7^2 \\ &\quad - 2 \cdot 6 \cdot 7 \cos 135^\circ \\ &= 85 + 42\sqrt{2}. \end{aligned}$$

So $s^2 = 85 + 42\sqrt{2}$, giving $a = 85$, $b = 42$, and $a + b = 127$.

Thus, the correct answer is **E**.

24. Let S be the set of all points (x, y) in the coordinate plane such that $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq \frac{\pi}{2}$. What is the area of the subset of S for which

$$\sin^2 x - \sin x \sin y + \sin^2 y \leq \frac{3}{4}?$$

A $\frac{\pi^2}{9}$

B $\frac{\pi^2}{8}$

C $\frac{\pi^2}{6}$

D $\frac{3\pi^2}{16}$

E $\frac{2\pi^2}{9}$

Solution:

Fixing y , solve $\sin^2 x - \sin x \sin y + \sin^2 y = \frac{3}{4}$ as a quadratic in $\sin x$:

$$\begin{aligned}\sin x &= \frac{1}{2} \sin y \\ &\quad \pm \frac{\sqrt{3}}{2} \cos y \\ &= \sin\left(y \pm \frac{\pi}{3}\right).\end{aligned}$$

Within S , $\sin x = \sin\left(y - \frac{\pi}{3}\right)$ gives the line $x = y - \frac{\pi}{3}$, while $\sin x = \sin\left(y + \frac{\pi}{3}\right)$ gives $x = y + \frac{\pi}{3}$ for $y \leq \frac{\pi}{6}$ and $x = -y + \frac{2\pi}{3}$ for $y \geq \frac{\pi}{6}$.

These lines split S into regions; testing the corners shows the inequality holds only in the middle band. Its area is

$$\begin{aligned}
 & \left(\frac{\pi}{2}\right)^2 - \frac{1}{2} \left(\frac{\pi}{3}\right)^2 \\
 & \quad - 2 \cdot \frac{1}{2} \left(\frac{\pi}{6}\right)^2 \\
 & = \frac{\pi^2}{6}.
 \end{aligned}$$

Thus, the correct answer is **C**.

25. A sequence $a_1, a_2, \dots$ of non-negative integers is defined by the rule $a_{n+2} = |a_{n+1} - a_n|$ for $n \geq 1$. If $a_1 = 999$, $a_2 < 999$, and $a_{2006} = 1$, how many different values of a_2 are possible?

A 165

B 324

C 495

D 499

E 660

Solution:

The rule gives $a_n \equiv a_{n+3} \pmod{2}$, so a_2 has the same parity as $a_{2006} = 1$; thus a_2 is odd.

Every term is a multiple of $\gcd(a_1, a_2)$, and $a_{2006} = 1$ forces $\gcd(999, a_2) = 1$. Since $999 = 3^3 \cdot 37$, we need a_2 not divisible by 3 or 37.

Among the odd integers in $[1, 998]$ there are 499; removing the 166 multiples of 3 and 13 multiples of 37, then adding back the 4 multiples of 111, leaves

$$499 - 166 - 13 + 4 = 324.$$

Each such a_2 works. For consecutive positive terms u, v , the update $(u, v) \mapsto (v, |v - u|)$ reduces their maximum within at most two steps. Since both initial terms are at most 999, some $a_N = 0$ occurs by $N \leq 1999$.

The gcd of each consecutive pair is invariant, so the equal terms immediately before that zero both equal $\gcd(999, a_2) = 1$. The sequence then cycles through 1, 1, 0. Finally, $2006 \equiv 2 \pmod{3}$, so a_{2006} has the same odd parity as a_2 ; in this cycle it must therefore be 1.

Thus, the correct answer is **B**.

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