2009 AMC 12A Solutions

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1. Kim's flight took off from Newark at $10{:}34$ am and landed in Miami at $1{:}18$ pm. Both cities are in the same time zone. If her flight took h hours and m minutes, with $0 \leq m < 60$, what is $h + m$?

- A 46
- B 47
- C 50
- D 53
- E 54

Solution:

From $10{:}34$ am to $11{:}00$ am is 26 minutes, from $11{:}00$ am to $1{:}00$ pm is 2 hours, and from $1{:}00$ pm to $1{:}18$ pm is 18 minutes.

So the flight lasted 2 hours and $26 + 18 = 44$ minutes. Thus $h = 2, m = 44$, and $h + m = 46$.

Thus, the correct answer is **A**.

2. Which of the following is equal to

$$1+\frac{1}{1+\frac{1}{1+1}}?$$

A $\frac{5}{4}$

B $\frac{3}{2}$

C $\frac{5}{3}$

D 2

E 3

Solution:

Starting inside, $1+1=2$, so $1+\frac{1}{2}=\frac{3}{2}$. Then $\frac{1}{\frac{3}{2}}=\frac{2}{3}$, and $1+\frac{2}{3}=\frac{5}{3}$.

Thus, the correct answer is **C**.

3. What number is one third of the way from $\frac{1}{4}$ to $\frac{3}{4}$?

A $\frac{1}{3}$

B $\frac{5}{12}$

C $\frac{1}{2}$

D $\frac{7}{12}$

E $\frac{2}{3}$

Solution:

The gap is $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. One third of the way adds $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

So the number is $\frac{1}{4} + \frac{1}{6} = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$.

Thus, the correct answer is **B**.

4. Four coins are picked out of a piggy bank that contains a collection of pennies, nickels, dimes, and quarters. Which of the following could not be the total value of the four coins, in cents?

A 15

B 25

C 35

D 45

E 55

Solution:

If the four coins include a penny, the total is not a multiple of 5, so it cannot equal any of the five listed multiples of 5. If there is no penny, every coin is worth at least 5 cents, so the total is at least 20 cents. Either way, 15 is impossible.

The other amounts are attainable: $25 = 10 + 5 + 5 + 5$, $35 = 10 + 10 + 10 + 5$, $45 = 25 + 10 + 5 + 5$, and $55 = 25 + 10 + 10 + 10$.

Thus, the correct answer is **A**.

5. One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

- A 8
- B 27
- C 64
- D 125
- E 216

Solution:

Let the cube have side length x. The new solid has dimensions $x+1, x-1$, and x, so its volume is $x(x+1)(x-1)=x^3-x$.

Setting this equal to x^3-5 gives $x^3-x=x^3-5$, so $x=5$.

The cube's volume is $5^3=125$.

Thus, the correct answer is **D**.

6. Suppose that $P = 2^m$ and $Q = 3^n$. Which of the following is equal to 12^{mn} for every pair of integers (m, n)?

A P^2Q

B P^nQ^m

C P^nQ^{2m}

D $P^{2m}Q^n$

E $P^{2n}Q^m$

Solution:

Since $12 = 2^2 \cdot 3$,

$$\begin{aligned} 12^{mn} &= 2^{2mn} \cdot 3^{mn} \\ &= (2^m)^{2n}(3^n)^m \\ &= P^{2n}Q^m. \end{aligned}$$

Thus, the correct answer is **E**.

7. The first three terms of an arithmetic sequence are $2x - 3, 5x - 11$, and $3x + 1$ respectively. The nth term of the sequence is 2009. What is n?

- A 255
- B 502
- C 1004
- D 1506
- E 8037

Solution:

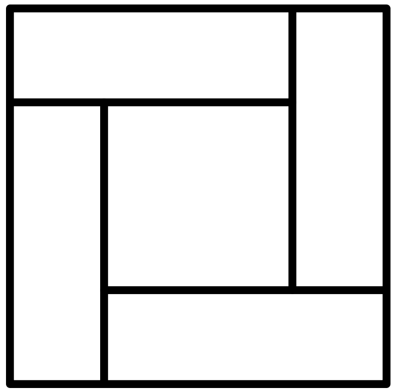
Equal consecutive differences give $(5x - 11) - (2x - 3) = (3x + 1) - (5x - 11)$, that is $3x - 8 = -2x + 12$, so $x = 4$.

The first three terms are $5, 9, 13$, with common difference 4.

The nth term satisfies $2009 = 5 + (n - 1) \cdot 4$, so $n - 1 = 501$ and $n = 502$.

Thus, the correct answer is **B**.

8. Four congruent rectangles are placed as shown. The area of the outer square is 4 times that of the inner square. What is the ratio of the length of the longer side of each rectangle to the length of its shorter side?



A 3

B $\sqrt{10}$

C $2+\sqrt{2}$

D $2\sqrt{3}$

E 4

Solution:

Let the rectangles have shorter side x and longer side y. The outer square has side $x+y$ and the inner square has side $y-x$.

Since the outer area is 4 times the inner area, the side ratio is $\sqrt{4}=2$, so $x+y=2(y-x)$.

This gives $y=3x$, so the ratio of longer to shorter side is 3.

Thus, the correct answer is **A**.

9. Suppose that $f(x+3) = 3x^2 + 7x + 4$ and $f(x) = ax^2 + bx + c$. What is $a + b + c$?

- A -1
- B 0
- C 1
- D 2
- E 3

Solution:

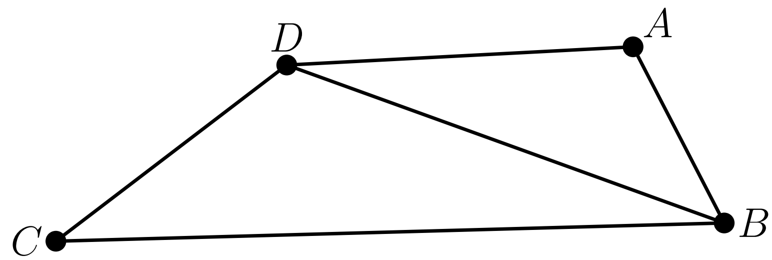
Note that $a + b + c = f(1)$.

Using $f(x+3) = 3x^2 + 7x + 4$ with $x = -2$ gives

$$\begin{aligned} f(1) &= f(-2+3) \\ &= 3(-2)^2 + 7(-2) + 4 \\ &= 12 - 14 + 4 = 2. \end{aligned}$$

Thus, the correct answer is **D**.

10. In quadrilateral $ABCD, AB = 5, BC = 17, CD = 5, DA = 9,$ and BD is an integer. What is BD?



A 11

B 12

C 13

D 14

E 15

Solution:

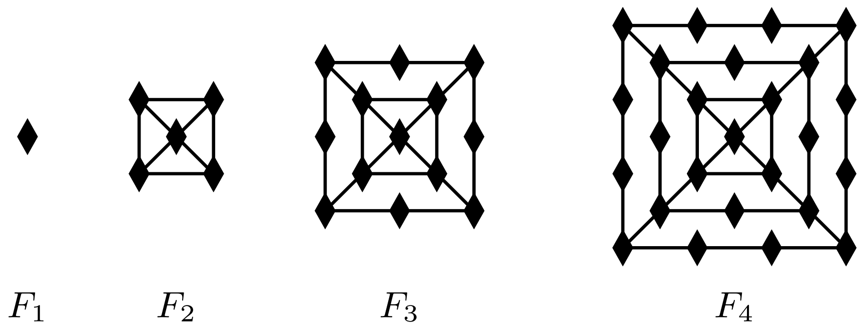
In $\triangle BCD,$ the triangle inequality gives $BD + CD > BC,$ so $BD + 5 > 17$ and $BD > 12.$

In $\triangle ABD, AB + DA > BD,$ so $BD < 5 + 9 = 14.$

The only integer with $12 < BD < 14$ is $13.$

Thus, the correct answer is **C**.

11. The figures F_1, F_2, F_3, and F_4 shown are the first in a sequence of figures. For $n \geq 3$, F_n is constructed from F_{n-1} by surrounding it with a square and placing one more diamond on each side of the new square than F_{n-1} had on each side of its outside square. For example, figure F_3 has 13 diamonds. How many diamonds are there in figure F_{20}?



A 401

B 485

C 585

D 626

E 761

Solution:

The outside square of F_n has 4 more diamonds than that of F_{n-1}, and the outside square of F_2 has 4, so the outside square of F_n has $4(n-1)$ diamonds.

Adding all the rings,

$$\begin{aligned} &1 + 4\big(1 + 2 + \cdots + (n-1)\big) \\ &= 1 + 4 \cdot \frac{(n-1)n}{2} \\ &= 1 + 2(n-1)n. \end{aligned}$$

For $n = 20$, this is $1 + 2 \cdot 19 \cdot 20 = 761.$

Thus, the correct answer is **E**.

12. How many positive integers less than 1000 are 6 times the sum of their digits?

A 0

B 1

C 2

D 4

E 12

Solution:

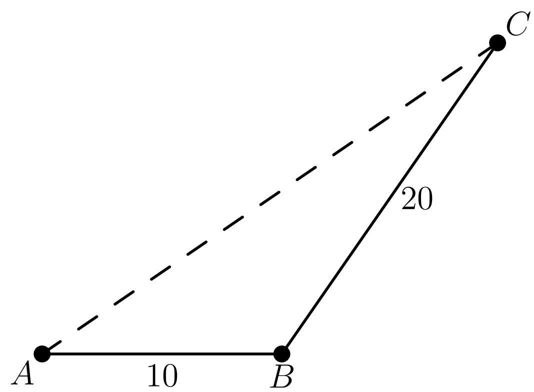
If $N = 6 \cdot (\text{digit sum})$, then since the digit sum of a number below 1000 is at most 27, we have $N \leq 162$.

For a two-digit number $10t + u = 6(t + u)$ gives $4t = 5u$, forcing $t = 5$ and $u = 4$, so $N = 54$. A one-digit number would need $6u = u$, impossible for $u > 0$. A three-digit number $100h + 10t + u = 6(h + t + u)$ gives $94h + 4t = 5u$, whose left side is at least 94 while the right side is at most 45, so there is no solution.

Hence exactly one number, 54, works.

Thus, the correct answer is **B**.

13. A ship sails 10 miles in a straight line from A to B, turns through an angle between 45° and 60°, and then sails another 20 miles to C. Let AC be measured in miles. Which of the following intervals contains AC^2?



A $[400, 500]$

B $[500, 600]$

C $[600, 700]$

D $[700, 800]$

E $[800, 900]$

Solution:

By the Law of Cosines,

$$\begin{aligned} AC^2 &= 10^2 + 20^2 \\ &\quad - 2 \cdot 10 \cdot 20 \cos(\angle ABC) \\ &= 500 - 400 \cos(\angle ABC). \end{aligned}$$

The ship turns through an angle between 45° and 60°, so the interior angle $\angle ABC$ lies between 120° and 135°.

Since $\cos 120^\circ = -\frac{1}{2}$ and $\cos 135^\circ = -\frac{\sqrt{2}}{2}$,

$$\begin{aligned} 700 &= 500 + 200 \\ &\leq AC^2 \leq 500 + 200\sqrt{2} \\ &< 800. \end{aligned}$$

So AC^2 lies in $[700, 800]$.

Thus, the correct answer is **D**.

14. A triangle has vertices $(0,0)$, $(1,1)$, and $(6m,0)$, and the line $y=mx$ divides the triangle into two triangles of equal area. What is the sum of all possible values of m?

A $-\frac{1}{3}$

B $-\frac{1}{6}$

C $\frac{1}{6}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

The line $y=mx$ passes through the vertex $(0,0)$, so it bisects the triangle's area exactly when it passes through the midpoint of the opposite side, joining $(1,1)$ and $(6m,0)$. That midpoint is $\left(\frac{6m+1}{2}, \frac{1}{2}\right)$.

Requiring it to satisfy $y=mx$ gives

$$\frac{1}{2} = m \cdot \frac{6m+1}{2},$$

so $6m^2+m-1=0$, that is $(3m-1)(2m+1)=0$.

The possible values are $m=\frac{1}{3}$ and $m=-\frac{1}{2}$, whose sum is $-\frac{1}{6}$.

Thus, the correct answer is **B**.

15. For what value of n is

$$
\begin{aligned}
&i + 2i^2 + 3i^3 + \cdots + ni^n \\
&= 48 + 49i?
\end{aligned}
$$

Note: here $i = \sqrt{-1}$.

A 24

B 48

C 49

D 97

E 98

Solution:

For k a multiple of 4,

$$
\begin{aligned}
&(k+1)i^{k+1} + (k+2)i^{k+2} \\
&\quad + (k+3)i^{k+3} + (k+4)i^{k+4} \\
&= (k+1)i - (k+2) \\
&\quad - (k+3)i + (k+4) \\
&= 2 - 2i.
\end{aligned}
$$

Summing the first 96 terms (that is 24 blocks) gives $24(2-2i) = 48 - 48i$.

Adding the next term $97i^{97} = 97i$ yields $48 - 48i + 97i = 48 + 49i$. So $n = 97$.

Thus, the correct answer is **D**.

16. A circle with center C is tangent to the positive x- and y-axes and externally tangent to the circle centered at $(3, 0)$ with radius 1. What is the sum of all possible radii of the circle with center C?

A 3

B 4

C 6

D 8

E 9

Solution:

A circle tangent to both positive axes with radius r has center (r, r). External tangency to the circle at $(3, 0)$ of radius 1 means the distance between centers is $r + 1$:

$$(r-3)^2 + r^2 = (r+1)^2.$$

Expanding gives $r^2 - 8r + 8 = 0$. Both roots $r = 4 \pm 2\sqrt{2}$ are positive, and by Vieta's formulas their sum is 8.

Thus, the correct answer is **D**.

17. Let $a + ar_1 + ar_1^2 + ar_1^3 + \cdots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $r_1 + r_2$?

A 0

B $\frac{1}{2}$

C 1

D $\frac{1+\sqrt{5}}{2}$

E 2

Solution:

For a series with first term a and ratio r, the sum is $\dfrac{a}{1-r} = r$, so $r^2 - r + a = 0$.

Both r_1 and r_2 satisfy this same quadratic, and since the two series are different, $r_1 \neq r_2$, so they are its two distinct roots. By Vieta's formulas, $r_1 + r_2 = 1$.

Thus, the correct answer is **C**.

18. For $k > 0$, let $I_k = 10\ldots064$, where there are k zeros between the 1 and the 6. Let $N(k)$ be the number of factors of 2 in the prime factorization of I_k. What is the maximum value of $N(k)$?

A 6

B 7

C 8

D 9

E 10

Solution:

Note that $I_k = 10^{k+2} + 64 = 2^{k+2}5^{k+2} + 2^6$.

For $k < 4$ the first term has fewer than 6 factors of 2, so $N(k) < 6$. For $k > 4$ the first term is divisible by 2^7 but the 2^6 term is not, so $N(k) < 7$.

For $k = 4, I_4 = 2^6(5^6 + 1)$. Since $5^6 + 1 = (5^2 + 1)\left((5^2)^2 - 5^2 + 1\right) = 26 \cdot 601,$ and $26 = 2 \cdot 13$ contributes exactly one more factor of 2, we get $N(4) = 7$.

So the maximum value is 7.

Thus, the correct answer is **B**.

19. Andrea inscribed a circle inside a regular pentagon, circumscribed a circle around the pentagon, and calculated the area of the region between the two circles. Bethany did the same with a regular heptagon (7 sides). The areas of the two regions were A and B, respectively. Each polygon had a side length of 2. Which of the following is true?

A $A = \frac{25}{49}B$

B $A = \frac{5}{7}B$

C $A = B$

D $A = \frac{7}{5}B$

E $A = \frac{49}{25}B$

Solution:

For a regular polygon with side length 2, let O be the center, M the midpoint of a side, and N an endpoint of that side. Then $\triangle OMN$ has a right angle at M, with $MN = 1$, $OM = r$ (inradius), and $ON = R$ (circumradius).

So $R^2 - r^2 = 1$, and the area between the circles is $\pi(R^2 - r^2) = \pi$ for any number of sides. Hence $A = B$.

Thus, the correct answer is **C**.

20. Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

A $\frac{9}{2}$

B $\frac{50}{11}$

C $\frac{21}{4}$

D $\frac{17}{3}$

E 6

Solution:

Adding $\triangle CED$ to each of $\triangle AED$ and $\triangle BEC$ shows $\triangle ACD$ and $\triangle BCD$ have equal areas. They share base CD, so A and B are equidistant from line CD, meaning $AB \parallel CD$.

Then $\triangle ABE \sim \triangle CDE$ with ratio $\dfrac{AB}{CD} = \dfrac{9}{12} = \dfrac{3}{4}$, so $\dfrac{AE}{EC} = \dfrac{3}{4}$.

Writing $AE = 3x$ and $EC = 4x$, we get $7x = AC = 14$, so $x = 2$ and $AE = 6$.

Thus, the correct answer is **E**.

21. Let $p(x) = x^3 + ax^2 + bx + c$, where $a, b,$ and c are complex numbers. Suppose that

$$p(2009 + 9002\pi i)$$
$$= p(2009)$$
$$= p(9002) = 0.$$

What is the number of nonreal zeros of $x^{12} + ax^8 + bx^4 + c$?

A 4

B 6

C 8

D 10

E 12

Solution:

Since $x^{12} + ax^8 + bx^4 + c = p(x^4)$, a value is a zero exactly when x^4 equals one of the roots of p, namely $2009 + 9002\pi i, 2009,$ or $9002.$

The equation $x^4 = 2009 + 9002\pi i$ has four distinct nonreal roots. Each of $x^4 = 2009$ and $x^4 = 9002$ has two real roots and two nonreal roots.

So the nonreal zeros number $4 + 2 + 2 = 8.$

Thus, the correct answer is **C**.

22. A regular octahedron has side length 1. A plane parallel to two of its opposite faces cuts the octahedron into two congruent solids. The polygon formed by the intersection of the plane and the octahedron has area $\frac{a\sqrt{b}}{c}$, where $a, b,$ and c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime. What is $a + b + c$?

A 10

B 11

C 12

D 13

E 14

Solution:

Let the two parallel faces be triangles. The plane passes through the midpoints of the six edges not on those faces, forming an equilateral hexagon of side $\frac{1}{2}$, which by symmetry is also equiangular and hence regular.

A regular hexagon is six equilateral triangles, so its area is

$$6 \cdot \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2 = \frac{3\sqrt{3}}{8}.$$

Thus $a = 3, b = 3, c = 8,$ and $a + b + c = 14.$

Thus, the correct answer is **E**.

23. Functions f and g are quadratic, $g(x) = -f(100 - x)$, and the graph of g contains the vertex of the graph of f. The four x-intercepts on the two graphs have x-coordinates x_1, x_2, x_3, and x_4, in increasing order, and $x_3 - x_2 = 150$. The value of $x_4 - x_1$ is $m + n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. What is $m + n + p$?

A 602

B 652

C 702

D 752

E 802

Solution:

Because $g(x) = -f(100 - x)$, the graphs of f and g are reflections of each other through the point $(50, 0)$, so the four intercepts pair up with $x_2 + x_3 = x_1 + x_4 = 100$.

With $x_3 - x_2 = 150$, we get $x_2 = -25$ and $x_3 = 125$.

Take x_1, x_3 as the roots of f, whose vertex has x-coordinate $h = \dfrac{x_1 + x_3}{2}$, so $x_1 = 2h - 125$. The condition that the vertex of f lies on the graph of g gives

$$\begin{aligned} 1 &= \frac{f(h)}{g(h)} \\ &= \frac{(125 - h)(h - 125)}{-(h + 25)(3h - 225)}, \end{aligned}$$

which gives $h = -25 \pm 75\sqrt{2}$. Since $x_1 = 2h - 125$ must be less than $x_2 = -25$, we need $h < 50$, so $h = -25 - 75\sqrt{2}$.

Then $x_4 = 100 - x_1$, so

$$\begin{aligned} x_4 - x_1 &= 350 - 4h \\ &= 450 + 300\sqrt{2}. \end{aligned}$$

Hence $m + n + p = 450 + 300 + 2 = 752.$

Thus, the correct answer is **D**.

24. The tower function of twos is defined recursively as follows: $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Let $A = (T(2009))^{T(2009)}$ and $B = (T(2009))^{A}$. What is the largest integer k such that

$$\underbrace{\log_2 \log_2 \log_2 \ldots \log_2 B}_{k}$$

is defined?

A 2009

B 2010

C 2011

D 2012

E 2013

Solution:

Since $\log_2 T(n+1) = T(n)$, each application of $\log_2$ strips one 2 off the top of a tower of twos.

Write $T_j = T(j)$, and let L_j be the result of applying $\log_2$ to B exactly j times. The first two results are

$$\begin{aligned} L_1 &= A\,T_{2008}, \\ L_2 &= T_{2009}T_{2008} + T_{2007}. \end{aligned}$$

For the lower bound,

$$\begin{aligned} L_3 &> \log_2(T_{2009}T_{2008}) \\ &= T_{2008} + T_{2007} \\ &> T_{2008}. \end{aligned}$$

Repeatedly taking logarithms gives $L_{k+3} > T_{2008-k}$ for $0 \leq k \leq 2007$. In particular, $L_{2010} > 2$, so $L_{2011} > 1$ and $L_{2012} > 0$. Thus L_{2013} is defined.

For the upper bound, $T_{2007} < T_{2008}T_{2009}$, so

$$\begin{aligned} L_3 &< 1 + T_{2007} + T_{2008} \\ &< 2T_{2008}, \\ L_4 &< 1 + T_{2007} < T_{2008}. \end{aligned}$$

Repeating the last comparison gives $L_{k+4} < T_{2008-k}$ for $0 \le k \le 2007.$ Hence $L_{2011} < 2.$ Together with the lower bound, this yields $0 < L_{2012} < 1,$ so $L_{2013} < 0.$ Therefore L_{2014} is undefined, and the largest possible k is $2013.$

Thus, the correct answer is **E**.

25. The first two terms of a sequence are $a_1 = 1$ and $a_2 = \frac{1}{\sqrt{3}}$. For $n \geq 1$,

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}.$$

What is $|a_{2009}|$?

A 0

B $2 - \sqrt{3}$

C $\frac{1}{\sqrt{3}}$

D 1

E $2 + \sqrt{3}$

Solution:

The recursion is exactly the tangent addition formula, and $a_1 = \tan\frac{\pi}{4}, a_2 = \tan\frac{\pi}{6}$.

Writing $a_n = \tan\frac{\pi c_n}{12}$ with $c_1 = 3, c_2 = 2$, and $c_{n+2} \equiv c_n + c_{n+1} \pmod{12}$, the sequence c_n is

$$\begin{gathered} 3, 2, 5, 7, 0, 7, 7, 2, \\ 9, 11, 8, 7, 3, 10, 1, 11, \\ 0, 11, 11, 10, 9, 7, 4, 11, \ldots \end{gathered}$$

which is periodic with period 24.

Since $2009 = 24 \cdot 83 + 17$, $c_{2009} = c_{17} = 0$, so $a_{2009} = \tan 0 = 0$ and $|a_{2009}| = 0$.

Thus, the correct answer is **A**.

Problems: https://live.poshenloh.com/past-contests/amc12/2009A