2015 AMC 12A Solutions

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1. What is the value of

$$(2^0 - 1 + 5^2 + 0)^{-1} \times 5?$$

A -125

B -120

C $\frac{1}{5}$

D $\frac{5}{24}$

E 25

Solution:

Inside the parentheses,

$$\begin{aligned} & 2^0 - 1 + 5^2 + 0 \\ & \quad = 1 - 1 + 25 + 0 = 25. \end{aligned}$$

Then

$$(25)^{-1} \times 5 = \frac{5}{25} = \frac{1}{5}.$$

Thus, the correct answer is **C**.

2. Two of the three sides of a triangle are 20 and 15. Which of the following numbers is not a possible perimeter of the triangle?

A 52

B 57

C 62

D 67

E 72

Solution:

By the Triangle Inequality, the third side s satisfies $20 - 15 < s < 20 + 15$, that is $5 < s < 35$.

The perimeter is $35 + s$, so it lies strictly between 40 and 70. Among the choices, only 72 falls outside this range.

Thus, the correct answer is **E**.

3. Mr. Patrick teaches math to 15 students. He was grading tests and found that when he graded everyone's test except Payton's, the average grade for the class was 80. After he graded Payton's test, the class average became 81. What was Payton's score on the test?

- A 81
- B 85
- C 91
- D 94
- E 95

Solution:

The sum of the 14 other scores was $14 \cdot 80 = 1120$. The sum of all 15 scores was $15 \cdot 81 = 1215$.

Therefore Payton's score was $1215 - 1120 = 95$.

Thus, the correct answer is **E**.

4. The sum of two positive numbers is 5 times their difference. What is the ratio of the larger number to the smaller number?

A $\frac{5}{4}$

B $\frac{3}{2}$

C $\frac{9}{5}$

D 2

E $\frac{5}{2}$

Solution:

Let x and y be the numbers with $x > y > 0$. Then $x + y = 5(x - y)$.

Expanding gives $x + y = 5x - 5y$, so $6y = 4x$ and

$$\frac{x}{y} = \frac{3}{2}.$$

Thus, the correct answer is **B**.

5. Amelia needs to estimate the quantity $\frac{a}{b} - c$, where $a, b,$ and c are large positive integers. She rounds each of the integers so that the calculation will be easier to do mentally. In which of these situations will her answer necessarily be greater than the exact value of $\frac{a}{b} - c$?

A She rounds all three numbers up.

B She rounds a and b up, and she rounds c down.

C She rounds a and c up, and she rounds b down.

D She rounds a up, and she rounds b and c down.

E She rounds c up, and she rounds a and b down.

Solution:

To make $\frac{a}{b}$ larger, round the numerator a up and the denominator b down. To make $-c$ larger, round c down.

Only choice (D) does all three: it rounds a up while rounding b and c down, so every change pushes the estimate above the exact value. In the other choices at least one change works the wrong way, so the estimate is not guaranteed to be larger.

Thus, the correct answer is **D**.

6. Two years ago Pete was three times as old as his cousin Claire. Two years before that, Pete was four times as old as Claire. In how many years will the ratio of their ages be $2 : 1$?

A 2

B 4

C 5

D 6

E 8

Solution:

Let p and c be Pete's and Claire's current ages. Then $p - 2 = 3(c - 2)$ and $p - 4 = 4(c - 4)$.

Solving these gives $p = 20$ and $c = 8$, so Pete is 12 years older than Claire.

The ratio is $2 : 1$ when Claire is 12, which is $12 - 8 = 4$ years from now.

Thus, the correct answer is **B**.

7. Two right circular cylinders have the same volume. The radius of the second cylinder is 10% more than the radius of the first. What is the relationship between the heights of the two cylinders?

A The second height is 10% less than the first.

B The first height is 10% more than the second.

C The second height is 21% less than the first.

D The first height is 21% more than the second.

E The second height is 80% of the first.

Solution:

Let r, h and R, H be the radii and heights of the first and second cylinders. The volumes are equal, so $\pi r^2 h = \pi R^2 H$, and $R = 1.1r$.

Then $\pi r^2 h = \pi(1.1r)^2 H = \pi(1.21r^2)H$. Dividing by πr^2 yields $h = 1.21H$, so the first height is 21% more than the second.

Thus, the correct answer is **D**.

8. The ratio of the length to the width of a rectangle is $4:3$. If the rectangle has diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k?

A $\frac{2}{7}$

B $\frac{3}{7}$

C $\frac{12}{25}$

D $\frac{16}{25}$

E $\frac{3}{4}$

Solution:

Let the sides of the rectangle be $4a$ and $3a$. By the Pythagorean Theorem the diagonal is $5a = d$, so $a = \frac{d}{5}$.

The area is $4a \cdot 3a = 12a^2 = 12\left(\frac{d}{5}\right)^2 = \frac{12}{25}d^2$, so $k = \frac{12}{25}$.

Thus, the correct answer is **C**.

9. A box contains 2 red marbles, 2 green marbles, and 2 yellow marbles. Carol takes 2 marbles from the box at random; then Claudia takes 2 of the remaining marbles at random; and then Cheryl takes the last 2 marbles. What is the probability that Cheryl gets 2 marbles of the same color?

A $\frac{1}{10}$

B $\frac{1}{6}$

C $\frac{1}{5}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

Because the marbles left for Cheryl are determined at random, her two marbles are equally likely to be any pair. Fixing her first marble, the second is equally likely to be any of the 5 remaining marbles.

Exactly one of those 5 matches the first marble in color, so the probability is $\frac{1}{5}$.

Thus, the correct answer is **C**.

10. Integers x and y with $x > y > 0$ satisfy $x + y + xy = 80$. What is x?

A 8

B 10

C 15

D 18

E 26

Solution:

Adding 1 to both sides and factoring gives

$$(x+1)(y+1) = 81 = 3^4.$$

Because x and y are distinct positive integers with $x > y$, the only possibility is $x + 1 = 3^3 = 27$ and $y + 1 = 3$. Therefore $x = 26$.

Thus, the correct answer is **E**.

11. On a sheet of paper, Isabella draws a circle of radius $2,$ a circle of radius $3,$ and all possible lines simultaneously tangent to both circles. Isabella notices that she has drawn exactly $k \geq 0$ lines. How many different values of k are possible?

A 2

B 3

C 4

D 5

E 6

Solution:

The number of common tangent lines depends on the relative position of the two circles:

If the smaller circle is inside the larger, there are 0 tangents. If it is internally tangent, there is $1.$ If the circles intersect at two points, there are $2.$ If they are externally tangent, there are $3.$ If they are separated, there are $4.$

Thus k can be any of $0, 1, 2, 3, 4,$ which gives 5 possible values.

Thus, the correct answer is **D**.

12. The parabolas $y = ax^2 - 2$ and $y = 4 - bx^2$ intersect the coordinate axes in exactly four points, and these four points are the vertices of a kite of area 12. What is $a + b$?

A 1

B 1.5

C 2

D 2.5

E 3

Solution:

The y-intercepts of the two parabolas are -2 and 4. To intersect the x-axis, the first parabola opens upward and the second opens downward, so their x-intercepts are $\pm t$ for some $t > 0$.

The kite has one diagonal of length $4 - (-2) = 6$ along the y-axis and the other of length $2t$. Its area is $\frac{1}{2} \cdot 6 \cdot 2t = 6t = 12$, so $t = 2$.

Thus the x-intercepts are ± 2. For the first parabola, $0 = a(2)^2 - 2$ gives $a = \frac{1}{2}$; for the second, $0 = 4 - b(2)^2$ gives $b = 1$. Therefore $a + b = 1.5$.

Thus, the correct answer is **B**.

13. A league with 12 teams holds a round-robin tournament, with each team playing every other team exactly once. Games either end with one team victorious or else end in a draw. A team scores 2 points for every game it wins and 1 point for every game it draws. Which of the following is not a true statement about the list of 12 scores?

A. There must be an even number of odd scores.

B. There must be an even number of even scores.

C. There cannot be two scores of 0.

D. The sum of the scores must be at least 100.

E. The highest score must be at least 12.

Solution:

Each of the 12 teams plays 11 games, so $\dfrac{12 \cdot 11}{2} = 66$ games are played, and each game adds 2 points to the list. The total of all scores is $66 \cdot 2 = 132.$

If every game is a draw, each team scores $11,$ so the highest score need not reach $12;$ thus statement (E) can fail. The other statements always hold: the sum $132 \geq 100,$ the sum being even forces an even number of odd scores and hence an even number of even scores, and two teams cannot both score 0 because their mutual game gives at least one of them a point.

Thus, the correct answer is **E**.

14. What is the value of a for which

$$\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1?$$

A 9

B 12

C 18

D 24

E 36

Solution:

By the change-of-base formula, $\dfrac{1}{\log_b a} = \log_a b$. Therefore

$$\begin{aligned} 1 &= \log_a 2 + \log_a 3 \\ &\quad + \log_a 4 = \log_a 24. \end{aligned}$$

It follows that $a = 24$.

Thus, the correct answer is **D**.

15. What is the minimum number of digits to the right of the decimal point needed to express the fraction

$$\frac{123456789}{2^{26}\cdot 5^4}$$

as a decimal?

A 4

B 22

C 26

D 30

E 104

Solution:

The numerator and denominator share no common factors. To write the fraction as a decimal, rewrite it with a power of 10 in the denominator; the smallest that works is 10^{26} :

$$\frac{123456789}{2^{26}\cdot 5^4} = \frac{123456789\cdot 5^{22}}{10^{26}}.$$

Since the numerator $123456789\cdot 5^{22}$ is not divisible by $10,$ the decimal has exactly 26 places after the point.

Thus, the correct answer is **C**.

16. Tetrahedron $ABCD$ has $AB = 5, AC = 3, BC = 4, BD = 4, AD = 3,$ and $CD = \frac{12}{5}\sqrt{2}$. What is the volume of the tetrahedron?

A $3\sqrt{2}$

B $2\sqrt{5}$

C $\frac{24}{5}$

D $3\sqrt{3}$

E $\frac{24}{5}\sqrt{2}$

Solution:

Triangles ABC and ABD are 3-4-5 right triangles with area 6 and common hypotenuse AB. Let E be the foot of the altitude from C to AB; then $CE = \frac{3 \cdot 4}{5} = \frac{12}{5}$. Likewise the altitude from D meets AB at the same point E with $DE = \frac{12}{5}$.

Triangle CDE has sides $\frac{12}{5}, \frac{12}{5}$, and $CD = \frac{12}{5}\sqrt{2}$, so it is an isosceles right triangle with the right angle at E. Thus $DE \perp CE$ and $DE \perp AB$, making DE perpendicular to the plane of ABC.

The tetrahedron's volume is $\frac{1}{3} \cdot [ABC] \cdot DE = \frac{1}{3} \cdot 6 \cdot \frac{12}{5} = \frac{24}{5}$.

Thus, the correct answer is **C**.

17. Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

A $\frac{47}{256}$

B $\frac{3}{16}$

C $\frac{49}{256}$

D $\frac{25}{128}$

E $\frac{51}{256}$

Solution:

There are $2^8 = 256$ equally likely outcomes. Count the arrangements of standers (heads) with no two adjacent around the circle of 8 seats, grouped by how many people stand.

The number of ways to choose k non-adjacent seats from a circle of n is $\frac{n}{n-k}\binom{n-k}{k}$. For $n = 8$ this gives

$$1,\ 8,\ 20,\ 16,\ 2$$

for $k = 0, 1, 2, 3, 4,$ and more than 4 standers is impossible without an adjacency.

The total is $1 + 8 + 20 + 16 + 2 = 47,$ so the probability is $\frac{47}{256}$.

Thus, the correct answer is **A**.

18. The zeros of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

A 7

B 8

C 16

D 17

E 18

Solution:

Let the integer zeros be p and q. By Vieta's formulas $p + q = a$ and $pq = 2a$, so $pq = 2(p+q)$. Rearranging gives

$$(p-2)(q-2) = 4.$$

The integer factor pairs of 4 are $(1,4), (2,2), (4,1), (-1,-4), (-2,-2), (-4,-1)$, which yield (p,q) pairs summing to $a = 9, 8, -1$, and 0.

The distinct possible values of a are $9, 8, 0, -1$, whose sum is 16.

Thus, the correct answer is **C**.

19. For some positive integers p, there is a quadrilateral $ABCD$ with positive integer side lengths, perimeter p, right angles at B and C, $AB = 2$, and $CD = AD$. How many different values of $p < 2015$ are possible?

A 30

B 31

C 61

D 62

E 63

Solution:

In every such quadrilateral $CD \geq AB$. Let E be the foot of the perpendicular from A to $\overline{CD}$; then $CE = 2$ and $AE = BC$. Let $x = AE$ and $y = DE$, so $AD = 2 + y$.

By the Pythagorean Theorem $x^2 + y^2 = (2 + y)^2$, so $x^2 = 4 + 4y$ and x is even. Writing $x = 2z$ gives $y = z^2 - 1$, and the perimeter is

$$x + 2y + 6 = 2z^2 + 2z + 4.$$

Increasing values $z = 1, 2, 3, \ldots$ give the required quadrilaterals with increasing perimeter. For $z = 31$ the perimeter is 1988, and for $z = 32$ it is 2116. Therefore there are 31 possible values of $p < 2015$.

Thus, the correct answer is **B**.

20. Isosceles triangles T and T' are not congruent but have the same area and the same perimeter. The sides of T have lengths $5, 5$, and 8, while those of T' have lengths a, a, and b. Which of the following numbers is closest to b?

A 3

B 4

C 5

D 6

E 8

Solution:

The altitude of T to its base of length 8 is $\sqrt{5^2-4^2}=3$, so T has area $\frac{1}{2}\cdot 8\cdot 3=12$ and perimeter 18.

For T' we need $2a+b=18$ and area $\frac{1}{4}b\sqrt{4a^2-b^2}=12$. Substituting $a=\frac{18-b}{2}$ and squaring leads to

$$(b-8)(b^2-b-8)=0.$$

Since T and T' are not congruent, $b\neq 8$, so $b^2-b-8=0$ and $b=\frac{1+\sqrt{33}}{2}$. Because $25<33<36$, this is between 3 and 3.5, so the closest integer is 3.

Thus, the correct answer is **A**.

21. A circle of radius r passes through both foci of, and exactly four points on, the ellipse with equation $x^2 + 16y^2 = 16$. The set of all possible values of r is an interval $[a, b)$. What is $a + b$?

A $5\sqrt{2} + 4$

B $\sqrt{17} + 7$

C $6\sqrt{2} + 3$

D $\sqrt{15} + 8$

E 12

Solution:

The ellipse $\dfrac{x^2}{16} + y^2 = 1$ has semi-axes 4 and 1, so $c^2 = 16 - 1 = 15$ and the foci are $(\pm\sqrt{15}, 0)$.

A circle through both foci has its center on the y-axis, say $(0, k)$, with radius $\sqrt{k^2 + 15}$. Its top point always lies outside the ellipse. For four intersection points, its bottom point $(0, k - \sqrt{k^2 + 15})$ must be below $y = -1$, which happens exactly when $0 \leq k < 7$.

As k ranges over $[0, 7)$, the radius $\sqrt{k^2 + 15}$ ranges over $[\sqrt{15}, 8)$, so $a + b = \sqrt{15} + 8$.

Thus, the correct answer is **D**.

22. For each positive integer n, let $S(n)$ be the number of sequences of length n consisting solely of the letters A and B, with no more than three As in a row and no more than three Bs in a row. What is the remainder when $S(2015)$ is divided by 12?

A 0

B 4

C 6

D 8

E 10

Solution:

Note $S(1) = 2, S(2) = 4, S(3) = 8$. Every valid sequence ends in a run of one, two, or three equal letters; removing that run leaves a valid sequence of length $n-1, n-2$, or $n-3$. Thus

$$S(n) = S(n-1) + S(n-2) \\ + S(n-3).$$

Modulo 3, the first 13 terms are

$$2, 1, 2, 2, 2, 0, 1, 0, 1, 2, 0, 0, 2.$$

The next three terms are $2, 1, 2$, which reproduce the initial state, so the recurrence repeats with period 13. Since $2015 = 13 \cdot 155$, it follows that $S(2015) \equiv S(13) \equiv 2 \pmod 3$.

Modulo 4, the terms repeat as $2, 0, 0, 2$, because the next three-term state after these four terms is again $(2, 0, 0)$. Thus the period is 4. As $2015 = 4 \cdot 503 + 3$, we have $S(2015) \equiv S(3) \equiv 0 \pmod 4$.

Writing $S(2015) = 4k$, the condition $4k \equiv 2 \pmod 3$ gives $k \equiv 2 \pmod 3$, so $S(2015) \equiv 8 \pmod{12}$.

Thus, the correct answer is **D**.

23. Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a-b\pi}{c}$, where $a, b,$ and c are positive integers and $\gcd(a,b,c)=1$. What is $a+b+c$?

A 59

B 60

C 61

D 62

E 63

Solution:

The second point is on the same side as the first with probability $\frac{1}{4}$, on the opposite side with probability $\frac{1}{4}$, and on an adjacent side with probability $\frac{1}{2}$.

Opposite sides: the distance is at least $1 \geq \frac{1}{2}$ always, probability 1.

Same side: for points $(a,0)$ and $(b,0)$, the condition $|a-b| \geq \frac{1}{2}$ has probability $\frac{1}{4}$.

Adjacent sides: for points $(a,0)$ and $(0,b)$, the condition $\sqrt{a^2+b^2} \geq \frac{1}{2}$ is the region outside a quarter-circle of radius $\frac{1}{2}$, with probability $1-\frac{1}{4}\pi\left(\frac{1}{2}\right)^2 = 1-\frac{\pi}{16}$.

The total probability is

$$\begin{aligned}\frac{1}{4}\cdot 1+\frac{1}{4}\cdot\frac{1}{4} \\ +\frac{1}{2}\left(1-\frac{\pi}{16}\right) \\ =\frac{26-\pi}{32}.\end{aligned}$$

Thus $a+b+c=26+1+32=59$.

Thus, the correct answer is **A**.

24. Rational numbers a and b are chosen at random among all rational numbers in the interval $[0,2)$ that can be written as fractions $\frac{n}{d}$ where n and d are integers with $1 \leq d \leq 5$. What is the probability that

$$(\cos(a\pi) + i\sin(b\pi))^4$$

is a real number?

A $\frac{3}{50}$

B $\frac{4}{25}$

C $\frac{41}{200}$

D $\frac{6}{25}$

E $\frac{13}{50}$

Solution:

There are 20 possible values for each of a and b. In reduced form, denominators $1,2,3,4,5$ contribute respectively $2,2,4,4,8$ values in $[0,2)$, for a total of 20.

Writing $x = \cos(a\pi)$ and $y = \sin(b\pi)$, the fourth power $(x+iy)^4$ is real if and only if $x = 0, y = 0$, or $x = \pm y$.

The case $x = 0$ means $a \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$, giving $2 \cdot 20 = 40$ pairs; the case $y = 0$ means $b \in \{0,1\}$, giving another 40 pairs, of which 4 were already counted.

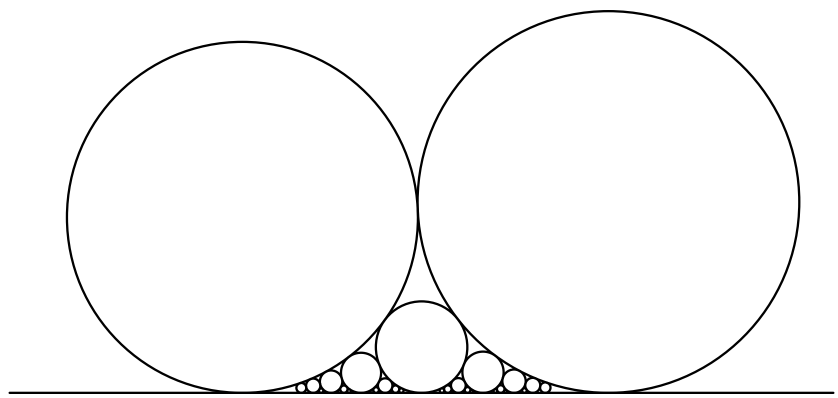
For the remaining condition $\cos(a\pi) = \pm\sin(b\pi)$, with neither side zero, the allowed values of b are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}$, and $\frac{7}{4}$. The corresponding numbers of allowed values of a are respectively $4,2,4,4,2,4$. For example, when $b = \frac{1}{4}$, the values are $a = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}$, and $\frac{7}{4}$; the other rows follow from the same quarter-turn identities. Hence this condition contributes $4+2+4+4+2+4 = 20$ more pairs.

In all there are $40 + 40 - 4 + 20 = 96$ valid pairs out of $400,$ so the probability is $\frac{96}{400} = \frac{6}{25}.$

Thus, the correct answer is **D**.

25. A collection of circles in the upper half-plane, all tangent to the x-axis, is constructed in layers as follows. Layer L_0 consists of two circles of radii 70^2 and 73^2 that are externally tangent. For $k \geq 1$, the circles in $\bigcup_{j=0}^{k-1} L_j$ are ordered according to their points of tangency with the x-axis. For every pair of consecutive circles in this order, a new circle is constructed externally tangent to each of the two circles in the pair. Layer L_k consists of the 2^{k-1} circles constructed in this way. Let $S = \bigcup_{j=0}^{6} L_j$, and for every circle C denote by $r(C)$ its radius. What is

$$\sum_{C \in S} \frac{1}{\sqrt{r(C)}}?$$



A $\frac{286}{35}$

B $\frac{583}{70}$

C $\frac{715}{73}$

D $\frac{143}{14}$

E $\frac{1573}{146}$

Solution:

If a circle of radius r is tangent to the x-axis and nestled in the crevice between two circles of radii r_1 and r_2 that are also tangent to the axis and to each other, then

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}.$$

Let $x = \frac{1}{\sqrt{70^2}} + \frac{1}{\sqrt{73^2}} = \frac{1}{70} + \frac{1}{73}$, which is the sum over L_0. The single circle of L_1 also contributes x. For $k \geq 2$, each new circle contributes the sum of its two neighbors, and every earlier circle is counted twice except the two circles of L_0; this yields a sum of $3^{k-1}x$ over L_k.

Therefore

$$\begin{aligned}\sum_{C \in S} \frac{1}{\sqrt{r(C)}} = x \\ + \sum_{k=1}^{6} 3^{k-1}x \\ = x\left(1 + \frac{3^6 - 1}{2}\right) \\ = x \cdot \frac{3^6 + 1}{2} \\ = 365x.\end{aligned}$$

Since $x = \frac{1}{70} + \frac{1}{73} = \frac{143}{70 \cdot 73} = \frac{143}{5110}$, the sum is $365 \cdot \frac{143}{5110} = \frac{143}{14}$.

Thus, the correct answer is **D**.

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