2019 AMC 12A Solutions

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1. The area of a pizza with radius 4 inches is N percent larger than the area of a pizza with radius 3 inches. What is the integer closest to N?

A 25

B 33

C 44

D 66

E 78

Solution:

The areas are proportional to the squares of the radii, so the ratio of the larger area to the smaller is $\frac{16}{9}$.

The percent increase is

$$\left(\frac{16}{9} - 1\right) \times 100 = \frac{700}{9} \approx 77.8.$$

The closest integer is 78.

Thus, the correct answer is **E**.

2. Suppose a is 150% of b. What percent of a is $3b$?

A 50

B $66\dfrac{2}{3}$

C 150

D 200

E 450

Solution:

Since $a = 1.5b$, we have

$$\frac{3b}{a} = \frac{3b}{1.5b} = 2.$$

As a percentage, $3b$ is 200% of a.

Thus, the correct answer is **D**.

3. A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

A 75

B 76

C 79

D 84

E 91

Solution:

In the worst case, we draw 14 each of red, green, and yellow, plus all of the blue (13), white (11), and black (9), without reaching 15 of any color.

That is

$$14 + 14 + 14 + 13 + 11 + 9 = 75$$

balls.

The next ball must complete a set of 15, so 76 balls are needed.

Thus, the correct answer is **B**.

4. What is the greatest number of consecutive integers whose sum is 45?

A 9

B 25

C 45

D 90

E 120

Solution:

Negative integers are allowed. The integers from -44 to 44 sum to 0, so the integers from -44 to 45 sum to 45.

This run has $45-(-44)+1=90$ integers. Conversely, if L consecutive integers have sum 45, then twice their sum is L times an integer, so $L \mid 90$. Hence $L \leq 90$, proving that this run is longest.

Thus, the correct answer is **D**.

5. Two lines with slopes $\dfrac{1}{2}$ and 2 intersect at $(2, 2)$. What is the area of the triangle enclosed by these two lines and the line $x + y = 10$?

A 4

B $4\sqrt{2}$

C 6

D 8

E $6\sqrt{2}$

Solution:

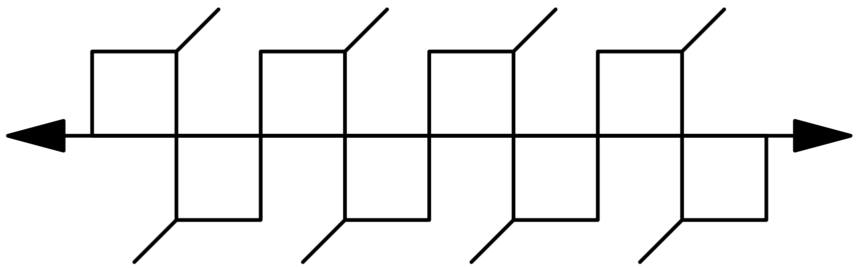
The two lines are $y = \frac{1}{2}x + 1$ and $y = 2x - 2$. Intersecting each with $x + y = 10$ gives the points $(6, 4)$ and $(4, 6)$.

The triangle has vertices $(2, 2)$, $(6, 4)$, and $(4, 6)$. By the shoelace formula,

$$\begin{aligned} &\tfrac{1}{2}\left|2(4-6) + 6(6-2) + 4(2-4)\right| \\ &= \tfrac{1}{2}\left|-4 + 24 - 8\right| \\ &= 6. \end{aligned}$$

Thus, the correct answer is **C**.

6. The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

A 0

B 1

C 2

D 3

E 4

Solution:

A translation by one full period maps the figure to itself, so translation works.

A 180° rotation about a suitable point on ℓ sends each square above the line to the square below it, with the diagonal segments matching, so this rotation works.

Reflection across ℓ sends the top-right diagonals to top-right diagonals below the line, but the actual below-line diagonals point to the bottom-left, so it fails. A reflection across a perpendicular line fails for the same reason. Only 2 of the four transformations work.

Thus, the correct answer is **C**.

7. Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, $\ldots$, 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

A $\mu < d < M$

B $M < d < \mu$

C $d = M = \mu$

D $d < M < \mu$

E $d < \mu < M$

Solution:

The values 1 through 28 each appear 12 times and are the modes, so $d = \frac{14+15}{2} = 14.5$.

The 183rd of the 365 ordered values is the median. Values 1 through 15 fill the first 180 positions, so position 183 is 16; thus $M = 16$.

The total of all values is $12(1 + \cdots + 28) + 11(29 + 30) + 7 \cdot 31 = 5738$, so $\mu = \frac{5738}{365} \approx 15.7$.

Therefore $d < \mu < M$.

Thus, the correct answer is **E**.

8. For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

A 14

B 16

C 18

D 19

E 21

Solution:

The values 0 and 1 occur when all four lines are parallel or all four are concurrent. Three parallel lines crossed by a fourth give 3 points. Three concurrent lines together with a fourth line not through their common point give 4. One parallel pair with no three concurrent gives 5, and four lines in general position give 6.

The value 2 is impossible. Once two lines meet at P, a third line not through P creates at least one new intersection; a fourth distinct line must then create another new point. If every remaining line passes through P, there is only one intersection point instead. Thus the achievable values are $0, 1, 3, 4, 5, 6$, whose sum is 19.

Thus, the correct answer is **D**.

9. A sequence of numbers is defined recursively by $a_1 = 1, a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

A 2020

B 4039

C 6057

D 6061

E 8078

Solution:

Taking reciprocals,

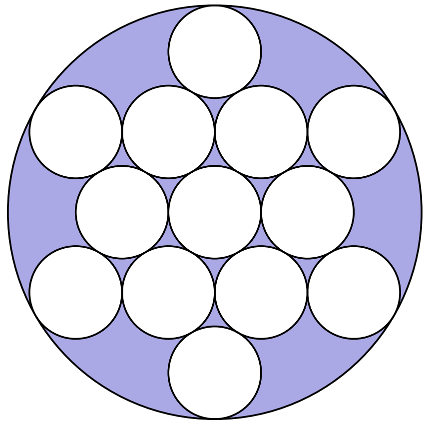
$$\begin{aligned} \frac{1}{a_n} &= \frac{2a_{n-2} - a_{n-1}}{a_{n-2}a_{n-1}} \\ &= \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}}. \end{aligned}$$

Let $b_n = \frac{1}{a_n}$. Then $b_n = 2b_{n-1} - b_{n-2}$, so b_n is arithmetic with $b_1 = 1, b_2 = \frac{7}{3}$, and common difference $\frac{4}{3}$.

Thus $b_{2019} = 1 + 2018 \cdot \frac{4}{3} = \frac{8075}{3}$, so $a_{2019} = \frac{3}{8075}$. Since these are relatively prime, $p + q = 8078.$

Thus, the correct answer is **E**.

10. The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?



A $4\pi\sqrt{3}$

B 7π

C $\pi(3\sqrt{3}+2)$

D $10\pi(\sqrt{3}-1)$

E $\pi(\sqrt{3}+6)$

Solution:

Place a unit circle at the center, six around it with centers at distance 2 (a hexagon), and six more with centers at distance $2\sqrt{3}$ in the outer gaps. That is $1+6+6=13$ circles.

The outermost circles are tangent to the big circle, whose radius is therefore $2\sqrt{3}+1$. Its area is

$$\pi(2\sqrt{3}+1)^2 = \pi(13+4\sqrt{3}).$$

Subtracting the 13 unit circles leaves $\pi(13+4\sqrt{3}) - 13\pi = 4\pi\sqrt{3}.$

Thus, the correct answer is **A**.

11. For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323\ldots_k$. What is k?

A 13

B 14

C 15

D 16

E 17

Solution:

The repeating block gives

$$0.\overline{23}_k = \frac{2k+3}{k^2-1} = \frac{7}{51}.$$

Cross-multiplying, $51(2k+3) = 7(k^2-1)$, so $7k^2 - 102k - 160 = 0$.

The quadratic formula gives

$$\begin{aligned} k &= \frac{102 + \sqrt{14884}}{14} \\ &= \frac{102+122}{14} = 16. \end{aligned}$$

Thus, the correct answer is **D**.

12. Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $\left(\log_2 \dfrac{x}{y}\right)^2$?

A $\dfrac{25}{2}$

B 20

C $\dfrac{45}{2}$

D 25

E 32

Solution:

Let $a = \log_2 x$ and $b = \log_2 y$. Then $\log_y 16 = \dfrac{4}{b}$, so $a = \dfrac{4}{b}$, giving $ab = 4$.

Since $xy = 64$, we have $a + b = 6$.

Therefore

$$\begin{aligned}\left(\log_2 \tfrac{x}{y}\right)^2 &= (a-b)^2 \\ &= (a+b)^2 - 4ab \\ &= 36 - 16 = 20.\end{aligned}$$

Thus, the correct answer is **B**.

13. How many ways are there to paint each of the integers $2, 3, \ldots, 9$ either red, green, or blue so that each number has a different color from each of its proper divisors?

A 144

B 216

C 256

D 384

E 432

Solution:

The primes 5 and 7 have no proper divisors here, giving 3 choices each.

Along the chain $2 \to 4 \to 8$, there are $3 \cdot 2 \cdot 1 = 6$ colorings. Number 9 must differ from 3, giving 2 choices once 3 is set.

Number 6 must differ from both 2 and 3. Summing over the colors of 2 and 3 (equal in 3 pairs, unequal in 6 pairs), the combined factor for $4, 8, 9, 6$ totals $2 \cdot 2 \cdot (3 \cdot 2 + 6 \cdot 1) = 48$.

Multiplying by the 9 ways for 5 and 7 gives $48 \cdot 9 = 432$.

Thus, the correct answer is **E**.

14. For a certain complex number c, the polynomial

$$P(x) = (x^2 - 2x + 2) \\ \cdot (x^2 - cx + 4) \\ \cdot (x^2 - 4x + 8)$$

has exactly 4 distinct roots. What is $|c|$?

A 2

B $\sqrt{6}$

C $2\sqrt{2}$

D 3

E $\sqrt{10}$

Solution:

The factors $x^2 - 2x + 2$ and $x^2 - 4x + 8$ have roots $1 \pm i$ and $2 \pm 2i$, which are 4 distinct values.

For P to have exactly 4 distinct roots, the roots of $x^2 - cx + 4$ must lie among these. Their product must equal 4, and the only such pair is one root from each factor, for example $(1 + i)(2 - 2i) = 4$.

Then $c = (1 + i) + (2 - 2i) = 3 - i$, so $|c| = \sqrt{3^2 + 1^2} = \sqrt{10}$.

Thus, the correct answer is **E**.

15. Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} \\ + \log \sqrt{a} + \log \sqrt{b} = 100$$

and all four terms on the left are positive integers, where $\log$ denotes the base 10 logarithm. What is ab?

A 10^{52}

B 10^{100}

C 10^{144}

D 10^{164}

E 10^{200}

Solution:

Let $\sqrt{\log a} = p$ and $\sqrt{\log b} = q$, so $\log a = p^2$ and $\log \sqrt{a} = \frac{p^2}{2}$. For this to be an integer, p is even; likewise q.

Writing $p = 2m, q = 2n$, the equation $p + q + \frac{p^2}{2} + \frac{q^2}{2} = 100$ becomes $m(m + 1) + n(n + 1) = 50.$

The only solution is $\{m, n\} = \{4, 5\}$, giving $\log(ab) = p^2 + q^2 = 4(16 + 25) = 164.$

Therefore $ab = 10^{164}.$

Thus, the correct answer is **D**.

16. The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

A $\frac{1}{21}$

B $\frac{1}{14}$

C $\frac{5}{63}$

D $\frac{2}{21}$

E $\frac{1}{7}$

Solution:

There are 5 odd and 4 even numbers. Each row and column must contain an odd number of odd entries.

The only way to place 5 odd entries with every row and column odd is to fill one complete row and one complete column (a plus shape of $3 + 3 - 1 = 5$ cells). There are $3 \cdot 3 = 9$ such patterns.

Each pattern admits $5!$ placements of the odd numbers and $4!$ of the even numbers, so the probability is

$$\frac{9 \cdot 5! \cdot 4!}{9!} = \frac{1}{14}.$$

Thus, the correct answer is **B**.

17. Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3, s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\, s_k + b\, s_{k-1} + c\, s_{k-2}$ for $k = 2, 3, \ldots$. What is $a + b + c$?

A -6

B 0

C 6

D 10

E 26

Solution:

Every root r satisfies $r^3 = 5r^2 - 8r + 13$, so $r^{k+1} = 5r^k - 8r^{k-1} + 13r^{k-2}$.

Summing over the three roots gives $s_{k+1} = 5s_k - 8s_{k-1} + 13s_{k-2}$, so $a = 5, b = -8, c = 13$.

Therefore $a + b + c = 5 - 8 + 13 = 10$.

Thus, the correct answer is **D**.

18. A sphere with center O has radius $6.$ A triangle with sides of length $15, 15,$ and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

A $2\sqrt{3}$

B 4

C $3\sqrt{2}$

D $2\sqrt{5}$

E 5

Solution:

The sphere intersects the triangle's plane in a circle of radius $\sqrt{36-d^2}$, where d is the distance from O to the plane. Since each side is tangent to the sphere, this circle is the triangle's incircle.

The triangle has area $\frac{1}{2}\cdot 24\cdot 9 = 108$ and semiperimeter $27,$ so its inradius is $\dfrac{108}{27} = 4.$

Thus $\sqrt{36-d^2} = 4,$ giving $d^2 = 20$ and $d = 2\sqrt{5}.$

Thus, the correct answer is **D**.

19. In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16},$$
$$\cos B = \frac{7}{8},$$
$$\cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

A 9

B 12

C 23

D 27

E 44

Solution:

Each sine is $\sqrt{1-\cos^2}$: $\sin A = \dfrac{3\sqrt{15}}{16}, \sin B = \dfrac{2\sqrt{15}}{16}, \sin C = \dfrac{4\sqrt{15}}{16}$.

By the Law of Sines the sides are in ratio $3:2:4$. The smallest integer sides are $3, 2, 4,$ which satisfy the triangle inequality.

The least perimeter is $3+2+4=9.$

Thus, the correct answer is **A**.

20. Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \dfrac{1}{2}$?

A $\dfrac{1}{3}$

B $\dfrac{7}{16}$

C $\dfrac{1}{2}$

D $\dfrac{9}{16}$

E $\dfrac{2}{3}$

Solution:

Each variable equals 0 with probability $\frac{1}{4}$, equals 1 with probability $\frac{1}{4}$, and is uniform on $[0, 1]$ with probability $\frac{1}{2}$.

Considering the nine combinations of types: the pairs $(0, 1)$ and $(1, 0)$ each contribute $\frac{1}{16}$. Each of the four point-versus-uniform cases contributes $\frac{1}{16}$. The uniform-versus-uniform case contributes $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

The total is $\dfrac{2 + 4 + 1}{16} = \dfrac{7}{16}$.

Thus, the correct answer is **B**.

21. Let

$$z = \frac{1+i}{\sqrt{2}}.$$

What is

$$\left(z^{1^2} + z^{2^2} + z^{3^2} + \dots + z^{12^2}\right) \cdot \left(\frac{1}{z^{1^2}} + \frac{1}{z^{2^2}} + \frac{1}{z^{3^2}} + \dots + \frac{1}{z^{12^2}}\right)?$$

A 18

B $72 - 36\sqrt{2}$

C 36

D 72

E $72 + 36\sqrt{2}$

Solution:

Since $z = e^{\frac{i\pi}{4}}$, we have $z^{k^2} = e^{\frac{i\pi k^2}{4}}$, depending only on $k^2 \bmod 8$.

For $k = 1$ to 12, the residue $k^2 \bmod 8$ is 1 (giving z) six times, 4 (giving -1) three times, and 0 (giving 1) three times. So the first sum is $6z - 3 + 3 = 6z$.

The second sum is likewise $\frac{6}{z} - 3 + 3 = \frac{6}{z}$. Their product is $6z \cdot \frac{6}{z} = 36$.

Thus, the correct answer is **C**.

22. Circles ω and γ, both centered at O, have radii 20 and 17, respectively. Equilateral triangle ABC, whose interior lies in the interior of ω but in the exterior of γ, has vertex A on ω, and the line containing side BC is tangent to γ. Segments AO and BC intersect at P, and $\dfrac{BP}{CP} = 3$. Then AB can be written in the form $\dfrac{m}{\sqrt{n}} - \dfrac{p}{\sqrt{q}}$ for positive integers m, n, p, q with $\gcd(m, n) = \gcd(p, q) = 1$. What is $m + n + p + q$?

A 42

B 86

C 92

D 114

E 130

Solution:

Let $s = AB$. Since $\dfrac{BP}{CP} = 3$, we have $BP = \dfrac{3s}{4}$ and $CP = \dfrac{s}{4}$. Put P at the origin with BC on the x-axis, $B = \left(-\frac{3s}{4}, 0\right)$, $C = \left(\frac{s}{4}, 0\right)$, and apex $A = \left(-\frac{s}{4}, \frac{s\sqrt{3}}{2}\right)$.

Points P, O, A are collinear, so $O = t \cdot A$ for some scalar t. Two conditions pin it down: O is at distance 17 from line BC, giving $|t| \cdot \dfrac{s\sqrt{3}}{2} = 17$, and A is on ω, giving $|t - 1| \cdot \dfrac{s\sqrt{13}}{4} = 20$ since $|A| = \dfrac{s\sqrt{13}}{4}$.

Solving, $|t|s = \dfrac{34}{\sqrt{3}}$ and $|t - 1|s = \dfrac{80}{\sqrt{13}}$. The valid configuration has O and A on opposite sides of P, so $|t - 1|s = |t|s + s$. Therefore

$$AB = s = \frac{80}{\sqrt{13}} - \frac{34}{\sqrt{3}}.$$

Then $m + n + p + q = 80 + 13 + 34 + 3 = 130$.

Thus, the correct answer is **E**.

23. Define binary operations $\diamondsuit$ and $\heartsuit$ by

$$a \diamondsuit b = a^{\log_7(b)}$$

and

$$a \heartsuit b = a^{\frac{1}{\log_7(b)}}$$

for all real numbers a and b for which these expressions are defined. The sequence (a_n) is defined recursively by $a_3 = 3 \heartsuit 2$ and

$$a_n = (n \heartsuit (n-1)) \diamondsuit a_{n-1}$$

for all integers $n \geq 4$. To the nearest integer, what is $\log_7(a_{2019})$?

A 8

B 9

C 10

D 11

E 12

Solution:

Let $L(x) = \log_7 x$. Then $L(a \diamondsuit b) = L(a)L(b)$ and $L(a \heartsuit b) = \dfrac{L(a)}{L(b)}$.

So $L(a_3) = \dfrac{L(3)}{L(2)}$, and $L(a_n) = \dfrac{L(n)}{L(n-1)} \cdot L(a_{n-1})$. The product telescopes:

$$\begin{aligned} L(a_N) &= \frac{L(3)}{L(2)} \cdot \frac{L(N)}{L(3)} \\ &= \frac{L(N)}{L(2)}. \end{aligned}$$

Hence $L(a_{2019}) = \dfrac{\log_7 2019}{\log_7 2} = \log_2 2019 \approx 10.98$, which rounds to 11.

Thus, the correct answer is **D**.

24. For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

A 31

B 32

C 33

D 34

E 35

Solution:

Fix a prime $p \le n$. By Legendre's formula, the difference between the exponent of p in the numerator and its exponent in the denominator is

$$D_p = \sum_{k\ge 1}\left\lfloor \frac{n^2-1}{p^k}\right\rfloor - n\sum_{k\ge 1}\left\lfloor \frac{n}{p^k}\right\rfloor .$$

If r_k is the remainder of n modulo p^k, the kth summand is $\left\lfloor \dfrac{nr_k-1}{p^k}\right\rfloor$.

Let $a = v_p(n)$. The first a summands are -1. If n is not a power of p, write $n = p^a m$ with $m \ge 2$. When $a \ge 1$, the next summand is at least $\dfrac{n}{p} - 1 \ge a$, and all later summands are nonnegative; when $a = 0$, every summand is already nonnegative. Thus $D_p \ge 0$ unless n is a power of p.

For $n = p^a$, Legendre's formula reduces the requirement to $p^a - 1 \ge 2a(p-1)$. Among prime powers at most 50, this fails exactly when $a = 1$ (so n is prime) and when $n = 2^2 = 4$. There are 15 primes at most 50, plus $n = 4$, giving 16 failures. Hence $50 - 16 = 34$ values of n work.

Thus, the correct answer is **D**.

25. Let $\triangle A_0B_0C_0$ be a triangle whose angle measures are exactly $59.999^\circ, 60^\circ$, and 60.001°. For each positive integer n define A_n to be the foot of the altitude from A_{n-1} to line $B_{n-1}C_{n-1}$. Likewise, define B_n to be the foot of the altitude from B_{n-1} to line $A_{n-1}C_{n-1}$, and C_n to be the foot of the altitude from C_{n-1} to line $A_{n-1}B_{n-1}$. What is the least positive integer n for which $\triangle A_nB_nC_n$ is obtuse?

A 10

B 11

C 13

D 14

E 15

Solution:

For an acute triangle, the orthic triangle (feet of the altitudes) has angles $180^\circ - 2\alpha$ for each original angle α.

Writing an angle as $60^\circ + x$, the new angle is $60^\circ - 2x$, so each deviation from 60° is multiplied by -2. The initial deviations are $\pm 0.001^\circ$.

After n steps a deviation has magnitude $0.001 \cdot 2^n$ degrees. The triangle first becomes obtuse when this exceeds 30°, i.e. $2^n > 30000$. Since $2^{14} = 16384$ and $2^{15} = 32768$, the least such n is 15.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2019A