2021 AMC 12A Fall Solutions

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1. What is the value of

$$\frac{(2112-2021)^2}{169}?$$

- A 7
- B 21
- C 49
- D 64
- E 91

Solution:

Since $2112-2021=91=7\cdot 13$ and $169=13^2$, the fraction is $\dfrac{(7\cdot 13)^2}{13^2}=7^2=49.$

Thus, the correct answer is **C**.

2. Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

A 16

B 17

C 18

D 19

E 20

Solution:

The original card is 4×6. Shortening a side by 1 inch gives area 18, which requires $3 \times 6 = 18$, so the reduced side was the 4-inch side.

Shortening the other side by 1 inch instead gives $4 \times 5 = 20$ square inches.

Thus, the correct answer is **E**.

3. Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

A $2\frac{3}{4}$

B $3\frac{3}{4}$

C $4\frac{1}{2}$

D $5\frac{1}{2}$

E $6\frac{3}{4}$

Solution:

Route A takes $\dfrac{6}{30} = \dfrac{1}{5}$ hour $= 12$ minutes.

Route B has 4.5 miles at 40 mph and 0.5 mile at 20 mph, taking $\dfrac{4.5}{40} + \dfrac{0.5}{20} = 0.1125 + 0.025 = 0.1375$ hour $= 8.25$ minutes.

The difference is $12 - 8.25 = 3.75 = 3\frac{3}{4}$ minutes.

Thus, the correct answer is **B**.

4. The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A?

A 1

B 3

C 5

D 7

E 9

Solution:

The number is $202100 + A$. Any even A makes it even, and $A = 5$ makes it divisible by 5, so A must be odd and not 5.

For $A = 1$ the digit sum is 6 (divisible by 3); for $A = 7$ the digit sum is 12 (divisible by 3); and $202103 = 11 \cdot 18373$. Only 202109 survives all tests, and it is prime.

Thus, the correct answer is **E**.

5. Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

A 6

B 8

C 10

D 11

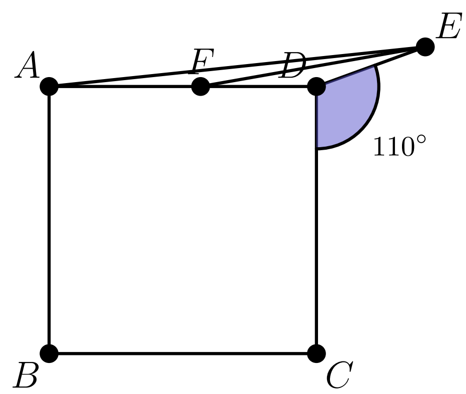
E 15

Solution:

There are 40 gaps between the first and 41st poles, so each gap is $\dfrac{5280}{40} = 132$ feet. Elmer's stride is $\dfrac{132}{44} = 3$ feet and Oscar's leap is $\dfrac{132}{12} = 11$ feet, a difference of $11 - 3 = 8$ feet.

Thus, the correct answer is **B**.

6. As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110°$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



A 160

B 164

C 166

D 170

E 174

Solution:

Because $ABCD$ is a square, $\angle ADC = 90°$. Since E and A lie on opposite sides of line CD, ray DE is swung past DC, so the angle of triangle DFE at D (with F on $\overline{AD}$) is $\angle FDE = 360° - (\angle ADC + \angle CDE) = 360° - (90° + 110°) = 160°$.

Since $DF = DE$, triangle DFE is isosceles with base angles $\angle DFE = \frac{180° - 160°}{2} = 10°$.

As A, F, D are collinear, $\angle AFE = 180° - \angle DFE = 170°$.

Thus, the correct answer is **D**.

7. A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are $50, 20, 20, 5,$ and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?

A -18.5

B -13.5

C 0

D 13.5

E 18.5

Solution:

The teacher average is $t = \dfrac{50 + 20 + 20 + 5 + 5}{5} = \dfrac{100}{5} = 20.$

The student average weights each class size by how many students are in it:

$$\begin{aligned} s &= \frac{50^2 + 20^2 + 20^2 + 5^2 + 5^2}{100} \\ &= \frac{2500 + 400 + 400 + 25 + 25}{100} \\ &= 33.5. \end{aligned}$$

So $t - s = 20 - 33.5 = -13.5.$

Thus, the correct answer is **B**.

8. Let M be the least common multiple of all the integers 10 through 30, inclusive. Let N be the least common multiple of $M, 32, 33, 34, 35, 36, 37, 38, 39,$ and 40. What is the value of $\frac{N}{M}$?

A 1

B 2

C 37

D 74

E 2886

Solution:

$M = \operatorname{lcm}(10, \ldots, 30)$ contains 2^4 (from 16), 3^3 (from 27), 5^2 (from 25), 7, and every prime up to 29.

Among $32, \ldots, 40$, the only new contributions are $32 = 2^5$, which raises the power of 2 from 2^4 to 2^5, and the new prime 37. Everything else factors into primes and powers already in M.

Therefore $\frac{N}{M} = 2 \cdot 37 = 74.$

Thus, the correct answer is **D**.

9. A right rectangular prism whose surface area and volume are numerically equal has edge lengths $\log_2 x$, $\log_3 x$, and $\log_4 x$. What is x?

A $2\sqrt{6}$

B $6\sqrt{6}$

C 24

D 48

E 576

Solution:

Let $a = \log_2 x, b = \log_3 x, c = \log_4 x$. Surface area equals volume gives $2(ab + bc + ca) = abc$. Dividing by abc,

$$1 = 2\left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right).$$

Since $\frac{1}{a} = \log_x 2$, etc., the sum is $\log_x 2 + \log_x 3 + \log_x 4 = \log_x 24$. Thus $1 = 2\log_x 24$, so $\log_x 24 = \frac{1}{2}$, meaning $x^{\frac{1}{2}} = 24$ and $x = 576$.

Thus, the correct answer is **E**.

10. The base-nine representation of the number N is $27{,}006{,}000{,}052_{\text{nine}}$. What is the remainder when N is divided by 5?

A 0

B 1

C 2

D 3

E 4

Solution:

Since $9 \equiv -1 \pmod 5$, each power $9^k \equiv (-1)^k$, so N is congruent to the alternating sum of its base-nine digits.

The nonzero digits, with their positions from the right, are 2 (position 0), 5 (position 1), 6 (position 6), 7 (position 9), and 2 (position 10). The alternating sum is $2 - 5 + 6 - 7 + 2 = -2 \equiv 3 \pmod 5$.

Thus, the correct answer is **D**.

11. Consider two concentric circles of radius 17 and 19. The larger circle has a chord, half of which lies inside the smaller circle. What is the length of the chord in the larger circle?

A $12\sqrt{2}$

B $10\sqrt{3}$

C $\sqrt{17 \cdot 19}$

D 18

E $8\sqrt{6}$

Solution:

Let the chord lie at distance d from the common center. Its total length is $2\sqrt{361 - d^2}$, and the portion inside the smaller circle has length $2\sqrt{289 - d^2}$.

Since half the chord lies inside, $2\sqrt{289 - d^2} = \frac{1}{2} \cdot 2\sqrt{361 - d^2}$. Squaring gives $4(289 - d^2) = 361 - d^2$, so $3d^2 = 795$ and $d^2 = 265$.

The chord length is $2\sqrt{361 - 265} = 2\sqrt{96} = 8\sqrt{6}$.

Thus, the correct answer is **E**.

12. What is the number of terms with rational coefficients among the 1001 terms in the expansion of

$$\left(x\sqrt[3]{2}+y\sqrt{3}\right)^{1000}?$$

A 0

B 166

C 167

D 500

E 501

Solution:

The general term is $\binom{1000}{k}(x\sqrt[3]{2})^{1000-k}(y\sqrt{3})^{k}$, whose coefficient contains $2^{\frac{1000-k}{3}}$ and $3^{\frac{k}{2}}$. This is rational exactly when $3 \mid (1000-k)$ and k is even.

Since $1000 \equiv 1 \pmod{3}$, we need $k \equiv 1 \pmod{3}$ and k even, which combine to $k \equiv 4 \pmod{6}$. The valid values $k = 4, 10, \ldots, 1000$ number $\dfrac{1000-4}{6}+1 = 167.$

Thus, the correct answer is **C**.

13. The angle bisector of the acute angle formed at the origin by the graphs of the lines $y = x$ and $y = 3x$ has equation $y = kx$. What is k?

A $\dfrac{1+\sqrt{5}}{2}$

B $\dfrac{1+\sqrt{7}}{2}$

C $\dfrac{2+\sqrt{3}}{2}$

D 2

E $\dfrac{2+\sqrt{5}}{2}$

Solution:

The bisector points along the sum of the unit vectors of the two lines: $\dfrac{(1,1)}{\sqrt{2}} + \dfrac{(1,3)}{\sqrt{10}}$.

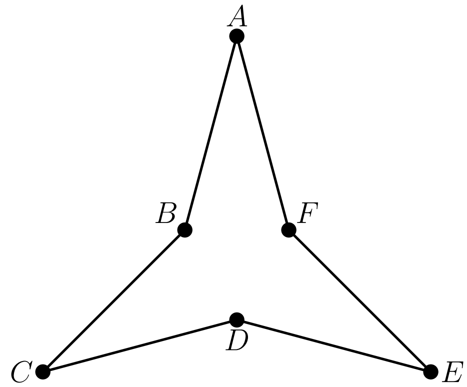
Its slope is

$$k = \frac{\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{10}}}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{10}}} = \frac{\sqrt{5}+3}{\sqrt{5}+1}.$$

Multiplying numerator and denominator by $\sqrt{5}-1$ gives $\dfrac{(\sqrt{5}+3)(\sqrt{5}-1)}{4} = \dfrac{2+2\sqrt{5}}{4} = \dfrac{1+\sqrt{5}}{2}$.

Thus, the correct answer is **A**.

14. In the figure, equilateral hexagon $ABCDEF$ has three nonadjacent acute interior angles that each measure 30°. The enclosed area of the hexagon is $6\sqrt{3}$. What is the perimeter of the hexagon?



A 4

B $4\sqrt{3}$

C 12

D 18

E $12\sqrt{3}$

Solution:

Let the common side length be s. The three acute vertices are the tips of isosceles triangles with two sides s and apex 30°; each has area $\frac{1}{2}s^2 \sin 30^\circ = \frac{s^2}{4}$.

The three reflex vertices form an inner equilateral triangle with side $2s \sin 15^\circ$, whose area is $\sqrt{3}\, s^2 \sin^2 15^\circ$. Using $\sin^2 15^\circ = \frac{2-\sqrt{3}}{4}$, the total area is

$$\frac{3s^2}{4} + \sqrt{3}\, s^2 \cdot \frac{2-\sqrt{3}}{4} = \frac{s^2\sqrt{3}}{2}.$$

Setting $\frac{s^2\sqrt{3}}{2} = 6\sqrt{3}$ gives $s^2 = 12$, so $s = 2\sqrt{3}$ and the perimeter is $6s = 12\sqrt{3}$.

Thus, the correct answer is **E**.

15. Recall that the conjugate of the complex number $w = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, is the complex number $\overline{w} = a - bi$. For any complex number z, let $f(z) = 4i\overline{z}$. The polynomial

$$P(z) = z^4 + 4z^3 + 3z^2 + 2z + 1$$

has four complex roots: z_1, z_2, z_3, and z_4. Let

$$Q(z) = z^4 + Az^3 + Bz^2 + Cz + D$$

be the polynomial whose roots are $f(z_1), f(z_2), f(z_3)$, and $f(z_4)$, where the coefficients A, B, C, and D are complex numbers. What is $B + D$?

A -304

B -208

C $12i$

D 208

E 304

Solution:

By Vieta on P, $\sum_{i<j} z_i z_j = 3$ and $\prod z_j = 1$, both real, so their conjugates are also 3 and 1.

The roots of Q are $4i\overline{z_j}$. Then B is the sum of products of pairs: $B = (4i)^2 \sum_{i<j} \overline{z_i}\,\overline{z_j}$ $= -16 \cdot 3 = -48$. And $D = (4i)^4 \prod \overline{z_j} = 256 \cdot 1 = 256$.

So $B + D = -48 + 256 = 208$.

Thus, the correct answer is **D**.

16. An organization has 30 employees, 20 of whom have a brand A computer while the other 10 have a brand B computer. For security, the computers can only be connected to each other and only by cables. The cables can only connect a brand A computer to a brand B computer. Employees can communicate with each other if their computers are directly connected by a cable or by relaying messages through a series of connected computers. Initially, no computer is connected to any other. A technician arbitrarily selects one computer of each brand and installs a cable between them, provided there is not already a cable between that pair. The technician stops once every employee can communicate with each other. What is the maximum possible number of cables used?

- A 190
- B 191
- C 192
- D 195
- E 196

Solution:

The technician keeps adding cables until the graph becomes connected. To maximize the count, keep the network disconnected for as long as possible: leave a single brand A computer isolated and fully connect the remaining 19 brand A computers to all 10 brand B computers.

That uses $19 \cdot 10 = 190$ cables while still disconnected. The next cable connects the last brand A computer, joining everyone, for a total of $190 + 1 = 191.$

Thus, the correct answer is **B**.

17. For how many ordered pairs (b, c) of positive integers does neither $x^2 + bx + c = 0$ nor $x^2 + cx + b = 0$ have two distinct real solutions?

A 4

B 6

C 8

D 12

E 16

Solution:

Neither quadratic has two distinct real roots exactly when both discriminants are nonpositive: $b^2 \leq 4c$ and $c^2 \leq 4b$.

Combining $c \geq \frac{b^2}{4}$ with $c \leq 2\sqrt{b}$ gives $b^{\frac{3}{2}} \leq 8$, so $b \leq 4$. Checking: $b = 1$ gives $c \in \{1, 2\}$; $b = 2$ gives $c \in \{1, 2\}$; $b = 3$ gives $c = 3$; and $b = 4$ gives $c = 4$.

That is $(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)$ – 6 ordered pairs.

Thus, the correct answer is **B**.

18. Each of 20 balls is tossed independently and at random into one of 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

A 1

B 4

C 8

D 12

E 16

Solution:

Both probabilities divide by 5^{20}, so $\frac{p}{q}$ is a ratio of arrangement counts.

For q, all bins have $4 : \frac{20!}{(4!)^5}$. For p, choose which bin has 3 and which has 5 in $5 \cdot 4 = 20$ ways, times $\frac{20!}{3!\,5!\,(4!)^3}$. Therefore

$$\begin{aligned}\frac{p}{q} &= 20 \cdot \frac{(4!)^5}{3!\,5!\,(4!)^3} \\ &= 20 \cdot \frac{(4!)^2}{3!\,5!} \\ &= 20 \cdot \frac{576}{720} \\ &= 16.\end{aligned}$$

Thus, the correct answer is **E**.

19. Let x be the least real number greater than 1 such that $\sin(x) = \sin(x^2)$, where the arguments are in degrees. What is x rounded up to the closest integer?

A 10

B 13

C 14

D 19

E 20

Solution:

Equal sines require $x^2 = x + 360k$ or $x^2 = 180 - x + 360k$ for some integer k.

The family $x^2 = x + 360k$ first exceeds 1 at $k = 1$, giving $x \approx 19.5$. The family $x^2 = 180 - x + 360k$ with $k = 0$ gives $x^2 + x - 180 = 0$, so $x = \dfrac{-1 + \sqrt{721}}{2} \approx 12.93$, which is smaller.

Rounded up, $x = 13$.

Thus, the correct answer is **B**.

20. For each positive integer n, let $f_1(n)$ be twice the number of positive integer divisors of n, and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$?

A 7

B 8

C 9

D 10

E 11

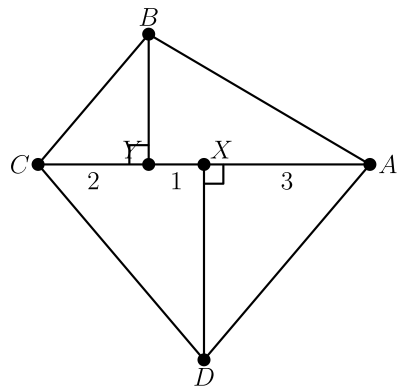
Solution:

Both 8 and 12 are fixed. For $n \leq 50$, the first value $f_1(n) = 2d(n)$ is at most 20. Checking the even values through 20 shows that an orbit reaches 12 exactly when its first value is $12, 18,$ or 20. Thus we need $d(n) = 6, 9,$ or 10.

The numbers at most 50 with 6 divisors are $12, 18, 20, 28, 32, 44, 45, 50$; the only one with 9 divisors is 36, and the only one with 10 divisors is 48. These 10 values all reach the fixed point 12.

Thus, the correct answer is **D**.

21. Let $ABCD$ be an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$ and $AB = CD$. Points X and Y lie on diagonal $\overline{AC}$ with X between A and Y, as shown in the figure. Suppose $\angle AXD = \angle BYC = 90^\circ$, $AX = 3$, $XY = 1$, and $YC = 2$. What is the area of $ABCD$?



A 15

B $5\sqrt{11}$

C $3\sqrt{35}$

D 18

E $7\sqrt{7}$

Solution:

Put $A = (0,0)$, $X = (3,0)$, $Y = (4,0)$, $C = (6,0)$. The right angles give $D = (3,t)$ and $B = (4,s)$ on opposite sides of AC.

Parallelism $\overline{AD} \parallel \overline{BC}$ forces $t = -\frac{3}{2}s$, and $AB = CD$ gives $16 + s^2 = 9 + t^2$, so $t^2 - s^2 = 7$. Substituting yields $s^2 = \frac{28}{5}$.

The shoelace formula gives area $= 3\,|t - s| = 3 \cdot \frac{5}{2}s = \frac{15}{2}s = \frac{15}{2}\sqrt{\frac{28}{5}} = 3\sqrt{35}$.

Thus, the correct answer is **C**.

22. Azar and Carl play a game of tic-tac-toe. Azar places an X in one of the boxes in a 3-by-3 array of boxes, then Carl places an O in one of the remaining boxes. After that, Azar places an X in one of the remaining boxes, and so on until all 9 boxes are filled or one of the players has 3 of their symbols in a row — horizontal, vertical, or diagonal — whichever comes first, in which case that player wins the game. Suppose the players make their moves at random, rather than trying to follow a rational strategy, and that Carl wins the game when he places his third O. How many ways can the board look after the game is over?

A 36

B 112

C 120

D 148

E 160

Solution:

Carl wins on his third O, so the board has three Os forming one of the 8 lines and three Xs in the other six cells. The Xs must not form a line (else Azar would have won first).

If the O line is a row or column (6 choices), the remaining six cells contain two full lines, so valid X placements number $\binom{6}{3} - 2 = 18$. If the O line is a diagonal (2 choices), the remaining six cells contain no full line, giving $\binom{6}{3} = 20$.

The total is $6 \cdot 18 + 2 \cdot 20 = 108 + 40 = 148.$

Thus, the correct answer is **D**.

23. A quadratic polynomial with real coefficients and leading coefficient 1 is called *disrespectful* if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

A $\frac{5}{16}$

B $\frac{1}{2}$

C $\frac{5}{8}$

D 1

E $\frac{9}{8}$

Solution:

Let p have roots r and s. Then $p(p(x)) = 0$ splits into $p(x) = r$ and $p(x) = s$, with discriminants $(r-s)^2 + 4r$ and $(r-s)^2 + 4s$. Exactly three real roots means one discriminant is 0 and the other positive.

Take $(r-s)^2 + 4s = 0$ and set $u = r - s$. Then $s = -\frac{u^2}{4}$ and the other discriminant is $4u$, so $u > 0$. Also $r + s = -\frac{u^2}{2} + u$, maximized at $u = 1$, giving $r = \frac{3}{4}$ and $s = -\frac{1}{4}$.

So $\tilde{p}(x) = \left(x - \frac{3}{4}\right)\left(x + \frac{1}{4}\right)$, and $\tilde{p}(1) = \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}$.

Thus, the correct answer is **A**.

24. Convex quadrilateral $ABCD$ has $AB = 18, \angle A = 60^\circ$, and $\overline{AB} \parallel \overline{CD}$. In some order, the lengths of the four sides form an arithmetic progression, and side AB is a side of maximum length. The length of another side is a. What is the sum of all possible values of a?

A 24

B 42

C 60

D 66

E 84

Solution:

Since $AB = 18$ is the largest, the four sides are $18, 18-d, 18-2d, 18-3d$. Placing $A = (0,0), B = (18,0)$, and $D = \left(\frac{m}{2}, \frac{m\sqrt{3}}{2}\right)$ with $m = DA$, let $n = CD$ and $\ell = BC$. Then $C = (\frac{m}{2} + n, \frac{m\sqrt{3}}{2})$, so

$$\ell^2 = m^2 + (18-n)^2 \\ - m(18-n).$$

Write $u_j = 18 - jd$ for $j = 1, 2, 3$. Substituting the permutations $123, 132, 213, 231, 312, 321$ of (u_1, u_2, u_3) for (m, n, ℓ) gives the nonzero candidates $d = 18, 2, 9, 5, 6, 6$, respectively. Positivity of u_3 requires $d < 6$, leaving only $d = 2$ and $d = 5$. The case $d = 0$ also gives a valid rhombus.

Thus the side sets are $\{18, 16, 14, 12\}, \{18, 13, 8, 3\}$, and $\{18, 18, 18, 18\}$.

The possible values of a non-AB side length are $\{3, 8, 12, 13, 14, 16, 18\}$, whose sum is 84.

Thus, the correct answer is **E**.

25. Let $m \geq 5$ be an odd integer, and let $D(m)$ denote the number of quadruples (a_1, a_2, a_3, a_4) of distinct integers with $1 \leq a_i \leq m$ for all i such that m divides $a_1 + a_2 + a_3 + a_4$. There is a polynomial

$$q(x) = c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

such that $D(m) = q(m)$ for all odd integers $m \geq 5$. What is c_1?

A -6

B -1

C 4

D 6

E 11

Solution:

Regard $1, 2, \ldots, m$ as all residues modulo m. Without the distinctness condition, the first three entries are arbitrary and the fourth is determined, giving m^3 ordered quadruples.

Inclusion-exclusion over equal-coordinate partitions gives the remaining terms. There are 6 choices of one equal pair, each leaving m^2 solutions, for $-6m^2$. The 3 two-pair partitions contribute $+3m$. The 4 triple-and-single partitions have inclusion-exclusion coefficient 2 and contribute $+8m$. Finally, the all-equal partition has coefficient -6 and one solution. (Here odd m makes 2 and 4 invertible modulo m.) Therefore

$$\begin{aligned} D(m) &= m^3 - 6m^2 \\ &\quad + 11m - 6 \\ &= (m-1)(m-2) \\ &\quad \cdot (m-3). \end{aligned}$$

Hence $c_1 = 11$.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2021C