2022 AMC 12A Solutions

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1. What is the value of

$$3+\cfrac{1}{3+\cfrac{1}{3+\frac{1}{3}}}?$$

- A $\dfrac{31}{10}$
- B $\dfrac{49}{15}$
- C $\dfrac{33}{10}$
- D $\dfrac{109}{33}$
- E $\dfrac{15}{4}$

Solution:

Simplify from the bottom. The innermost fraction is $3+\dfrac{1}{3}=\dfrac{10}{3}$.

The next layer is $3+\dfrac{1}{\frac{10}{3}}=3+\dfrac{3}{10}=\dfrac{33}{10}$.

Finally, $3+\dfrac{1}{\frac{33}{10}}=3+\dfrac{10}{33}=\dfrac{109}{33}$.

Thus, the correct answer is **D**.

2. The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

Let the third number be t. Then the first is $6t$ and the second is $t+40$. Their sum is

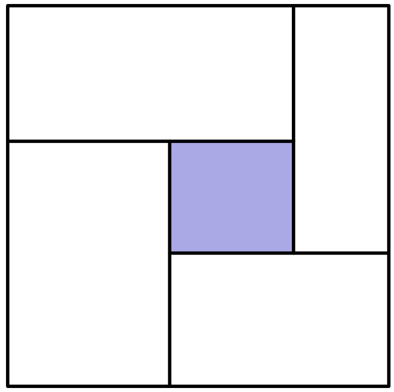
$$6t+(t+40)+t=8t+40=96,$$

so $t=7$.

The first number is 42 and the second is 47, so the difference has absolute value 5.

Thus, the correct answer is **E**.

3. Five rectangles, $A, B, C, D,$ and $E,$ are arranged in a square as shown below. These rectangles have dimensions $1 \times 6, 2 \times 4, 5 \times 6, 2 \times 7,$ and $2 \times 3,$ respectively. (The figure is not drawn to scale.) Which of the five rectangles is the shaded one in the middle?



- A A
- B B
- C C
- D D
- E E

Solution:

The five areas are $6,\ 8,\ 30,\ 14,$ and $6,$ which sum to $64.$ So the square is $8 \times 8.$

Placing C (5×6) across the top left, D (2×7) up the right side, E (2×3) in the lower left, and A (1×6) along the bottom leaves a central 2×4 gap, which is exactly rectangle $B.$

Thus, the correct answer is **B**.

4. The least common multiple of a positive integer n and 18 is $180,$ and the greatest common divisor of n and 45 is $15.$ What is the sum of the digits of $n?$

- A 3
- B 6
- C 8
- D 9
- E 12

Solution:

Since $180 = 2^2 \cdot 3^2 \cdot 5$ and $18 = 2 \cdot 3^2$, the condition $\operatorname{lcm}(n, 18) = 180$ forces n to contribute 2^2 and 5, with its power of 3 at most $2.$

From $\gcd(n, 45) = \gcd(n, 3^2 \cdot 5) = 15 = 3 \cdot 5$, the power of 3 in n is exactly 1 and the power of 5 is at least $1.$

Therefore $n = 2^2 \cdot 3 \cdot 5 = 60$, whose digits sum to $6.$

Thus, the correct answer is **B**.

5. Let the *taxicab distance* between points (x_1, y_1) and (x_2, y_2) in the coordinate plane be given by $|x_1 - x_2| + |y_1 - y_2|$. For how many points P with integer coordinates is the taxicab distance between P and the origin less than or equal to 20?

A 441

B 761

C 841

D 921

E 924

Solution:

For each $k \geq 1$, the set $|x| + |y| = k$ contains exactly $4k$ lattice points, and $k = 0$ gives the single origin.

The total is

$$1 + \sum_{k=1}^{20} 4k = 1 + 4 \cdot \frac{20 \cdot 21}{2}$$
$$= 1 + 840 = 841.$$

Thus, the correct answer is **C**.

6. A data set consists of 6 (not distinct) positive integers: $1, 7, 5, 2, 5,$ and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X?

A 10

B 26

C 32

D 36

E 40

Solution:

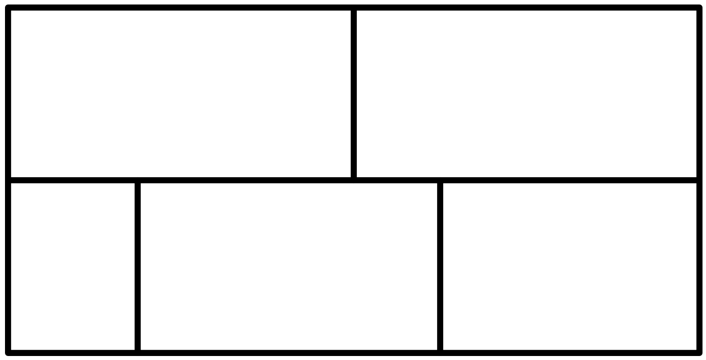
The known numbers sum to 20, so the mean is $\frac{20+X}{6}$, which must equal an element of the set.

Setting it to 5 gives $X = 10$; to 7 gives $X = 22$; and to X itself gives $20 + X = 6X$, so $X = 4$. Values 1 and 2 give negative X.

The positive values are $10, 22, 4$, summing to 36.

Thus, the correct answer is **D**.

7. A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



A 120

B 270

C 360

D 540

E 720

Solution:

The bottom-middle region shares a border with all four other regions. Color it first in 5 ways.

The top-left region borders it, giving 4 choices. Each of the three remaining regions borders exactly two already-colored regions, which have different colors, leaving 3 choices apiece.

The total is $5 \cdot 4 \cdot 3 \cdot 3 \cdot 3 = 540.$

Thus, the correct answer is **D**.

8. The infinite product

$$\sqrt[3]{10} \cdot \sqrt[3]{\sqrt[3]{10}} \cdot \sqrt[3]{\sqrt[3]{\sqrt[3]{10}}} \cdots$$

evaluates to a real number. What is that number?

A $\sqrt{10}$

B $\sqrt[3]{100}$

C $\sqrt[4]{1000}$

D 10

E $10\sqrt[3]{10}$

Solution:

The kth factor is 10 raised to the k-fold cube root, namely $10^{\frac{1}{3^k}}$.

The product is 10 raised to

$$\begin{aligned}\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots &= \frac{\frac{1}{3}}{1 - \frac{1}{3}} \\ &= \frac{1}{2}.\end{aligned}$$

So the value is $10^{\frac{1}{2}} = \sqrt{10}$.

Thus, the correct answer is **A**.

9. On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

A 7

B 12

C 21

D 27

E 31

Solution:

To "Are you a truth-teller?" the truth-tellers and liars both answer yes, and only alternaters who lie on this question answer yes. To "Are you an alternater?" the liars answer yes, and among alternaters only those telling the truth on this question answer yes. To "Are you a liar?" only alternaters lying on this question answer yes.

Split the alternaters by first response. Those starting with a lie answer (lie, truth, lie), so they say yes to all three questions; those starting truthful answer (truth, lie, truth) and say yes to none of the three. The 9 yeses on the last question are exactly the lie-first alternaters, so there are 9 of them.

The second question's 15 yeses are the liars plus these 9, so there are 6 liars. The first question's 22 yeses are truth-tellers plus liars plus the 9, so the truth-tellers number $22 - 6 - 9 = 7$.

Truth-tellers answer yes only to the first question, receiving one candy each, for $7 \cdot 1 = 7$ pieces.

Thus, the correct answer is **A**.

10. What is the number of ways the numbers from 1 to 14 can be split into 7 pairs such that for each pair, the greater number is at least 2 times the smaller number?

A 108

B 120

C 126

D 132

E 144

Solution:

Any number 8 or larger cannot be a smaller element (its double exceeds 14), so 8–14 are all larger elements and 1–7 are all smaller elements.

Match each smaller s to a larger $g \geq 2s$. Processing from the most restrictive: $s = 7$ forces $g = 14$ (1 way); then $s = 6$ has $\{12, 13\}$ left (2); $s = 5$ has 3; $s = 4$ has 4; $s = 3$ has 3; $s = 2$ has 2; $s = 1$ has 1.

The number of matchings is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 144$.

Thus, the correct answer is **E**.

11. What is the product of all real numbers x such that the distance on the number line between $\log_6 x$ and $\log_6 9$ is twice the distance on the number line between $\log_6 10$ and 1?

A 10

B 18

C 25

D 36

E 81

Solution:

The right-hand distance is $|\log_6 10 - 1| = \log_6 \frac{5}{3}$, so twice it is $2\log_6 \frac{5}{3} = \log_6 \frac{25}{9}$.

Thus $\left|\log_6 \frac{x}{9}\right| = \log_6 \frac{25}{9}$, giving $\frac{x}{9} = \frac{25}{9}$ or $\frac{x}{9} = \frac{9}{25}$, so $x = 25$ or $x = \frac{81}{25}$.

Their product is $25 \cdot \frac{81}{25} = 81$.

Thus, the correct answer is **E**.

12. Let M be the midpoint of $\overline{AB}$ in regular tetrahedron $ABCD$. What is $\cos(\angle CMD)$?

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{2}{5}$

D $\frac{1}{2}$

E $\frac{\sqrt{3}}{2}$

Solution:

Take edge length 1. Since M is the midpoint of $\overline{AB}$, segments CM and DM are altitudes of the equilateral faces, each of length $\frac{\sqrt{3}}{2}$. Also $CD = 1$.

By the Law of Cosines in $\triangle CMD$,

$$\begin{aligned}\cos(\angle CMD) &= \frac{\frac{3}{4} + \frac{3}{4} - 1}{2 \cdot \frac{3}{4}} \\ &= \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}.\end{aligned}$$

Thus, the correct answer is **B**.

13. Let $\mathcal{R}$ be the region in the complex plane consisting of all complex numbers z that can be written as the sum of complex numbers z_1 and z_2, where z_1 lies on the segment with endpoints 3 and $4i$, and z_2 has magnitude at most 1. What integer is closest to the area of $\mathcal{R}$?

A 13

B 14

C 15

D 16

E 17

Solution:

Adding a disk of radius 1 to every point of the segment sweeps out all points within distance 1 of it. The segment from 3 to $4i$ has length $\sqrt{3^2+4^2}=5.$

This "stadium" is a 5×2 rectangle plus two half-disks of radius 1, with area

$$5 \cdot 2 + \pi(1)^2 = 10 + \pi \approx 13.14.$$

The closest integer is $13.$

Thus, the correct answer is **A**.

14. What is the value of

$$(\log 5)^3 + (\log 20)^3 \\ + (\log 8)(\log 0.25)$$

where $\log$ denotes the base-ten logarithm?

A $\frac{3}{2}$

B $\frac{7}{4}$

C 2

D $\frac{9}{4}$

E 3

Solution:

Let $u = \log 2$. Then $\log 5 = 1 - u$, $\log 20 = 1 + u$, $\log 8 = 3u$, and $\log 0.25 = -2u$.

With $a = 1 - u,\ b = 1 + u$, we have $a + b = 2$ and $ab = 1 - u^2$, so

$$\begin{aligned} & a^3 + b^3 \\ &= (a+b)\big((a+b)^2 - 3ab\big) \\ &= 2\big(4 - 3(1 - u^2)\big) \\ &= 2 + 6u^2. \end{aligned}$$

The last term is $(3u)(-2u) = -6u^2$, so the total is $2 + 6u^2 - 6u^2 = 2$.

Thus, the correct answer is **C**.

15. The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

A $\frac{24}{5}$

B $\frac{42}{5}$

C $\frac{81}{5}$

D 30

E 48

Solution:

Let the roots be r, s, t. By Vieta's formulas, $r + s + t = \frac{39}{10}$, $rs + rt + st = \frac{29}{10}$, and $rst = \frac{6}{10} = \frac{3}{5}$.

The new volume is

$$\begin{aligned}
&(r+2)(s+2)(t+2) \\
&= rst + 2(rs + rt + st) \\
&\quad + 4(r + s + t) + 8 \\
&= \frac{3}{5} + \frac{58}{10} \\
&\quad + \frac{156}{10} + 8 \\
&= 30.
\end{aligned}$$

Thus, the correct answer is **D**.

16. A *triangular number* is a positive integer that can be expressed in the form $t_n = 1 + 2 + 3 + \cdots + n$, for some positive integer n. The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2$, $t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

A 6

B 9

C 12

D 18

E 27

Solution:

If $t_n = y^2$, then $\frac{n(n+1)}{2} = y^2$, or $(2n+1)^2 - 8y^2 = 1$. The positive Pell solutions occur successively by multiplying $(2n+1) + y\sqrt{8}$ by $3 + \sqrt{8}$.

Starting from $(2n+1, y) = (3, 1)$, this gives $(17, 6)$, $(99, 35)$, and then $(577, 204)$. Thus the fourth value has $n = 288$ and equals $204^2 = 41616$.

The sum of its digits is $4 + 1 + 6 + 1 + 6 = 18$.

Thus, the correct answer is **D**.

17. Suppose a is a real number such that the equation

$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a can be written in the form $(p, q) \cup (q, r)$, where p, q, and r are real numbers with $p < q < r$. What is $p + q + r$?

A -4

B -1

C 0

D 1

E 4

Solution:

Since $\sin x \neq 0$ on $(0, \pi)$, divide by $\sin x$:

$$\begin{aligned} a(1 + 2\cos x) &= 4\cos^2 x - 1 \\ &= (2\cos x - 1)(2\cos x + 1). \end{aligned}$$

When $\cos x = -\frac{1}{2}$ (that is, $x = \frac{2\pi}{3}$) both sides vanish, so this is a solution for every a. Otherwise we may cancel $1 + 2\cos x$ to get $a = 2\cos x - 1$, i.e. $\cos x = \dfrac{a+1}{2}$.

This yields a second solution in $(0, \pi)$ exactly when $-1 < \dfrac{a+1}{2} < 1$, that is $a \in (-3, 1)$, and it is distinct from $x = \frac{2\pi}{3}$ unless $a = -2$.

So more than one solution occurs for $a \in (-3, -2) \cup (-2, 1)$, giving $p + q + r = -3 - 2 + 1 = -4$.

Thus, the correct answer is **A**.

18. Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \ldots, T_n$ returns the point $(1, 0)$ back to itself?

A 359

B 360

C 719

D 720

E 721

Solution:

Rotating a point at angle θ by k° gives $\theta + k$, and reflecting across the y-axis sends angle ϕ to $180 - \phi$. So T_k sends θ to $(180 - k) - \theta$.

Starting from $(1, 0)$ at angle 0, applying $T_1, T_2, \ldots$ gives angles $179, -1, 178, -2, 177, \ldots$. After an even number $2m$ of steps the angle is $-m$, and after an odd number $2m + 1$ it is $179 - m$.

For the point to return, the angle must be a multiple of 360°. The even case needs $m = 360$, i.e. $n = 720$. The odd case needs $179 - m = 0$, i.e. $m = 179$ and $n = 359$, where the net reflection fixes $(1, 0)$.

The least such n is 359.

Thus, the correct answer is **A**.

19. Suppose that 13 cards numbered $1, 2, 3, \ldots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards $1, 2, 3$ are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, $7, 8, 9, 10$ on the fourth pass, and $11, 12, 13$ on the fifth pass. For how many of the $13!$ possible orderings of the cards will the 13 cards be picked up in exactly two passes?

7	11	8	6	4	5	9	12	1	13	10	2	3

A 4082

B 4095

C 4096

D 8178

E 8191

Solution:

Let $\mathrm{pos}(k)$ be the position of card k. A fresh pass is needed exactly when $\mathrm{pos}(k+1) < \mathrm{pos}(k)$, so the number of passes is one more than the number of descents in the sequence $\mathrm{pos}(1), \mathrm{pos}(2), \ldots, \mathrm{pos}(13)$.

To build a permutation with at most one descent, choose the entries before the possible descent and write both chosen blocks in increasing order. There are 2^{13} subsets. The 14 initial segments $\varnothing, \{1\}, \ldots, \{1, \ldots, 13\}$ produce no descent; every other subset produces a unique permutation with one descent. Hence the count is $2^{13} - 14 = 8178$.

Thus, the correct answer is **D**.

20. Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3,$ and $PD = 4$. What is $\frac{BC}{AD}$?

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution:

Place the trapezoid symmetric about the y-axis: $A = (-p, 0), D = (p, 0), B = (-q, h), C = (q, h)$, with $P = (x, y)$.

Then $PA^2 - PD^2 = 4px = 1 - 16 = -15$ and $PB^2 - PC^2 = 4qx = 4 - 9 = -5$. Dividing gives $\frac{p}{q} = 3$.

Since $AD = 2p$ and $BC = 2q$, we get $\frac{BC}{AD} = \frac{q}{p} = \frac{1}{3}$.

Thus, the correct answer is **B**.

21. Let $P(x) = x^{2022} + x^{1011} + 1$. Which of the following polynomials divides $P(x)$?

A $x^2 - x + 1$

B $x^2 + x + 1$

C $x^4 + 1$

D $x^6 - x^3 + 1$

E $x^6 + x^3 + 1$

Solution:

If ζ is a root of a divisor, then $P(\zeta) = \zeta^{2022} + \zeta^{1011} + 1 = 0$ is required.

The roots of $x^6 + x^3 + 1$ are the primitive 9th roots of unity, so $\zeta^9 = 1$. Reducing exponents modulo $9, 2022 \equiv 6$ and $1011 \equiv 3$, giving $\zeta^6 + \zeta^3 + 1$. Here $\omega = \zeta^3$ is a primitive cube root of unity, so this equals $\omega^2 + \omega + 1 = 0$.

The other four options fail: substituting their roots yields nonzero values (for instance, the primitive cube roots of unity give $P = 3$).

Thus, the correct answer is **E**.

22. Let c be a real number, and let z_1, z_2 be the two complex numbers satisfying the quadratic $z^2 - cz + 10 = 0$. Points $z_1, z_2, \dfrac{1}{z_1}$, and $\dfrac{1}{z_2}$ are the vertices of a (convex) quadrilateral Q in the complex plane. When the area of Q obtains its maximum value, c is the closest to which of the following?

A 4.5

B 5

C 5.5

D 6

E 6.5

Solution:

If the roots are non-real, then $z_1 = \sqrt{10}\, e^{i\theta}$ and $z_2 = \overline{z_1}$, since $z_1 z_2 = 10$. Then $\dfrac{1}{z_1} = \dfrac{1}{\sqrt{10}} e^{-i\theta}$ and $\dfrac{1}{z_2} = \dfrac{1}{\sqrt{10}} e^{i\theta}$.

The two vertical sides of this trapezoid have lengths $2\sqrt{10}\sin\theta$ and $\frac{2\sin\theta}{\sqrt{10}}$, and their horizontal separation is $(\sqrt{10} - \frac{1}{\sqrt{10}})\cos\theta$. Hence its area is

$$\frac{99}{10}\sin\theta\cos\theta = \frac{99}{20}\sin 2\theta,$$

which is maximized at $\theta = 45^\circ$.

Then $c = z_1 + z_2 = 2\sqrt{10}\cos 45^\circ = 2\sqrt{5} \approx 4.47$, closest to 4.5.

Thus, the correct answer is **A**.

23. Let h_n and k_n be the unique relatively prime positive integers such that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\frac{h_n}{k_n}.$$

Let L_n denote the least common multiple of the numbers $1, 2, 3, \ldots, n$. For how many integers n with $1 \le n \le 22$ is $k_n < L_n$?

A 0

B 3

C 7

D 8

E 10

Solution:

Always $k_n \mid L_n$, so $k_n < L_n$ exactly when some prime p divides both L_n and the numerator $N = \sum_{k=1}^{n} \frac{L_n}{k}$ (i.e. a prime cancels).

For a prime p with maximal power $p^a \le n$, only the terms with $v_p(k) = a$ keep p out of $\frac{L_n}{k}$; all others are divisible by p. So p cancels iff $\sum_{v_p(k)=a} \frac{L_n}{k} \equiv 0 \pmod{p}$.

Applying this test recursively with $H_n = H_{n-1} + \frac{1}{n}$ gives $k_n = L_n$ for $1 \le n \le 5$, $k_n = \frac{L_n}{3}$ for $6 \le n \le 8$, and $k_n = L_n$ again for $9 \le n \le 17$. Finally, the ratios $\frac{L_n}{k_n}$ for $n = 18, 19, 20, 21, 22$ are $3, 3, 15, 45, 45$, respectively. Thus cancellation occurs precisely for $n = 6, 7, 8, 18, 19, 20, 21, 22$, which is 8 values.

Thus, the correct answer is **D**.

24. How many strings of length 5 formed from the digits $0, 1, 2, 3, 4$ are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j? (For example, 02214 satisfies the condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

A 500

B 625

C 1089

D 1199

E 1296

Solution:

Sort the five digits as $d_{(1)} \leq d_{(2)} \leq \cdots \leq d_{(5)}$. The requirement "at least j digits less than j" is equivalent to $d_{(j)} \leq j - 1$ for $j = 1, 2, 3, 4$, i.e. $d_{(1)} = 0,\ d_{(2)} \leq 1,\ d_{(3)} \leq 2,\ d_{(4)} \leq 3$ (with $d_{(5)} \leq 4$ automatic).

These strings are exactly the parking functions of length 5. To count them, arrange 6 parking spaces in a circle and let 5 labeled cars choose arbitrary preferred spaces. Each car moves forward to the first open space. Among the 6^5 preference strings, rotating all preferences cycles the unique empty space through all 6 positions. Therefore exactly $\frac{6^5}{6} = 6^4 = 1296$ strings leave a specified space empty. Choosing that space as the extra sixth space gives precisely the sorted inequalities above.

Thus, the correct answer is **E**.

25. A circle with integer radius r is centered at (r, r). Distinct line segments of length c_i connect points $(0, a_i)$ to $(b_i, 0)$ for $1 \leq i \leq 14$ and are tangent to the circle, where a_i, b_i, and c_i are all positive integers and $c_1 \leq c_2 \leq \cdots \leq c_{14}$. What is the ratio $\frac{c_{14}}{c_1}$ for the least possible value of r?

A $\frac{21}{5}$

B $\frac{85}{13}$

C 7

D $\frac{39}{5}$

E 17

Solution:

The circle centered (r, r) with radius r is tangent to both axes. A segment from $(0, a)$ to $(b, 0)$ with $a^2 + b^2 = c^2$ is tangent to it when r equals either the inradius $\frac{a+b-c}{2}$ or the semiperimeter $\frac{a+b+c}{2}$ of the right triangle with legs a, b.

In the inradius case, put $x = a - 2r$ and $y = b - 2r$. Then $xy = 2r^2$, and every positive divisor x determines one oriented segment, with $a = x + 2r$, $b = \frac{2r^2}{x} + 2r$, and $c = x + \frac{2r^2}{x} + 2r$. Thus there are exactly $d(2r^2)$ such segments.

For $r = 1, 2, 3, 4, 5$, these counts are $2, 4, 6, 6, 6$, and no semiperimeter case is possible because the smallest integer right triangle has semiperimeter 6. At $r = 6$, $d(72) = 12$, and the 3-4-5 triangle contributes two more oriented segments. Hence 6 is the least possible radius and gives exactly 14 segments.

The two semiperimeter segments have $c = 5$. In the inradius family, $c = x + \frac{72}{x} + 12$ is largest at $x = 1$ or 72, giving $c = 85$. Therefore $c_1 = 5$, $c_{14} = 85$, and $\frac{c_{14}}{c_1} = 17$.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2022A