2018 AMC 8 Solutions

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https://live.poshenloh.com/past-contests/amc8/2018/solutions

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1. An amusement park has a collection of scale models, with a ratio of $1 : 20$, of buildings and other sights from around the country. The height of the United States Capitol is 289 feet. What is the height in feet of its replica at this park, rounded to the nearest whole number?

A 14

B 15

C 16

D 18

E 20

Solution(s):

Let the height of the replica be h. Since the ratio of the scale model to the real world is $1 : 20$, we know that

$$h : 289 = 1 : 20$$

Therefore:

$$\frac{h}{289} = \frac{1}{20}$$
$$h = \frac{289}{20}$$
$$h = 14.45 \approx 14$$

Thus, the correct answer is **A**.

2. What is the value of the product

$$(1+\frac{1}{1})\cdot(1+\frac{1}{2})\cdot(1+\frac{1}{3})$$
$$\cdot(1+\frac{1}{4})\cdot(1+\frac{1}{5})\cdot(1+\frac{1}{6})?$$

A $\frac{7}{6}$

B $\frac{4}{3}$

C $\frac{7}{2}$

D 7

E 8

Solution(s):

Let's first note that if we are given an expression of the form $1+\frac{1}{n}$, we can rewrite this as

$$\frac{n}{n}+\frac{1}{n}=\frac{n+1}{n}.$$

With that in mind, we can rewrite the expression given to us in the problem, as shown below:

$$\begin{aligned}&(1+\frac{1}{1})\cdot(1+\frac{1}{2})\cdot(1+\frac{1}{3})\\ &\cdot(1+\frac{1}{4})\cdot(1+\frac{1}{5})\cdot(1+\frac{1}{6})\\ &=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5}\cdot\frac{7}{6}\\ &=7\end{aligned}$$

Thus, the correct answer is **D**.

3. Students Arn, Bob, Cyd, Dan, Eve, and Fon are arranged in that order in a circle. They start counting: Arn first, then Bob, and so forth. When the number contains a 7 as a digit (such as 47) or is a multiple of 7 that person leaves the circle and the counting continues. Who is the last one present in the circle?

A Arn

B Bob

C Cyd

D Dan

E Eve

Solution(s):

Notice that the first 5 numbers that contains a 7 as its digit or are a multiple of 7 are $7, 14, 17, 21, 27$. Any player who lands on one of these numbers must leave the circle.

With this in mind, let's start counting. Initially, we have all 6 people, starting with A(Arn). After everyone says a number, A must say 7, so he leaves the circle.

The circle now has 5 members: B (Bob), C (Cyd), D (Dan), E (Eve), and F (Fon) -- with B restarting his counting at 8. Everyone in the circle counts without incident, and it loops around such that Bob says 13. However, this leaves Cyd to say 14, and he leaves the circle.

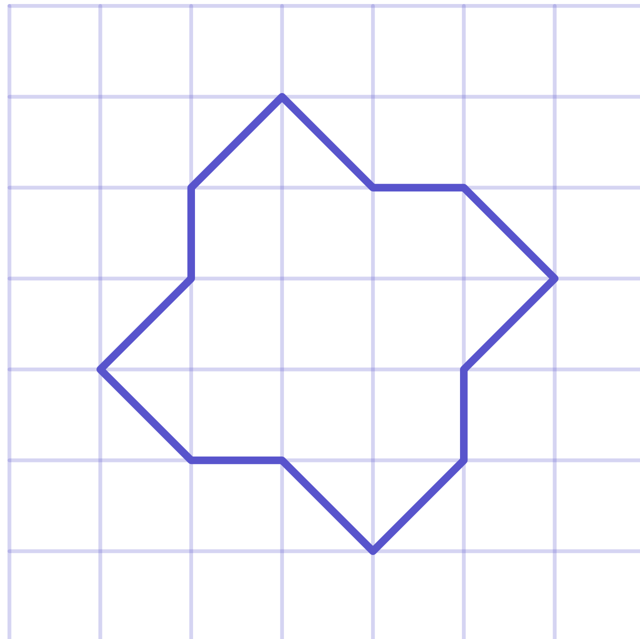
The circle now has 4 members: B, D, E, F -- with D continuing the counting at 15. E says 16, and F says 17, and therefore leaves the circle.

The circle now has 3 members: B, D, E -- with B continuing the counting at 18. The counting loops around, and B says 21, and therefore leaves the circle.

The circle now has 2 members: D, E -- with D starting at 22. They go back and forth, with D saying even numbers and E saying odd numbers. As such, eventually, E must say 27, and as such, leaves the circle. This makes D -- Dan -- the last one left in the circle.

Thus, **D** is the correct answer.

4. The twelve-sided figure shown has been drawn on $1 \text{ cm} \times 1 \text{ cm}$ graph paper. What is the area of the figure in cm^2?



A 12

B 12.5

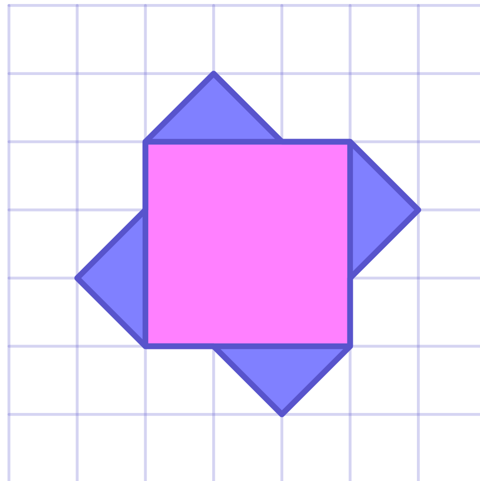
C 13

D 13.5

E 14

Solution(s):

To solve for the area of the figure, we seperate the compound shape into parts that are easier to work with, as such:



As is now clear, there is the center 3×3 square (in pink), with 4 smaller triangles surrounding it (in blue).

The area of the square is $3 \cdot 3 = 9$. The other triangles each have a base of 2 and a height of 1, so their area is equal to

$$\frac{bh}{2} = \frac{2 \cdot 1}{2} = 1.$$

There are 4 of these triangles, so their total area is $1 \cdot 4 = 4$.

Therefore, the total area is $9 + 4 = 13$.

Thus, the correct answer is **C**.

5. What is the value of

$$\begin{gathered}1+3+5+\cdots+2017+2019\\-2-4-6-\cdots-2016-2018?\end{gathered}$$

A -1010

B -1009

C 1008

D 1009

E 1010

Solution(s):

Rearranging the terms, notice that the expression in the question is equal to:

$$\begin{gathered}1+(3-2)+(5-4)+\cdots+\\(2017-2016)+(2019-2018).\end{gathered}$$

Each term is equal to 1, and there are $\frac{2019-1}{2}+1=1010$ terms, so the total sum is $1010\cdot 1=1010.$

Thus, **E** is the correct answer.

6. On a trip to the beach, Anh traveled 50 miles on the highway and 10 miles on a coastal access road. He drove three times as fast on the highway as on the coastal road. If Anh spent 30 minutes driving on the coastal road, how many minutes did his entire trip take?

A 50

B 70

C 80

D 90

E 100

Solution(s):

Anh drove 10 miles on the coastal road in 30 minutes. Therefore, his speed on the coastal road (notated as v_c) is:

$$\begin{aligned} v_c &= \frac{10\text{miles}}{30\text{minutes}} \\ &= \frac{1\text{mile}}{3\text{minutes}} \\ &= \frac{1}{3}\frac{\text{miles}}{\text{minute}}. \end{aligned}$$

Since he drives 3 times as fast on highway (i.e. $v_h = 3v_c$), he drives at a speed of

$$3 \cdot \frac{1}{3}\frac{\text{miles}}{\text{minute}} = 1\frac{\text{miles}}{\text{minute}}$$

on the highway. Armed with these two facts, we know that Anh drove for 30 minutes on the coastal road, and he drove 50 miles at 1 mile per minute. This means it takes 50 minutes to drive the 50 miles on the highway.

As such, the total travel time is $50 + 30 = 80$ minutes.

Thus, the correct answer is **C**.

7. The 5-digit number $\underline{2}\ \underline{0}\ \underline{1}\ \underline{8}\ \underline{U}$ is divisible by 9. What is the remainder when this number is divided by 8?

- A 1
- B 3
- C 5
- D 6
- E 7

Solution(s):

Notice that a number is divisible by 9 if and only if the sum of its digits is also divisible by 9.

The sum of the digits of the 5-digit number in the problem is:

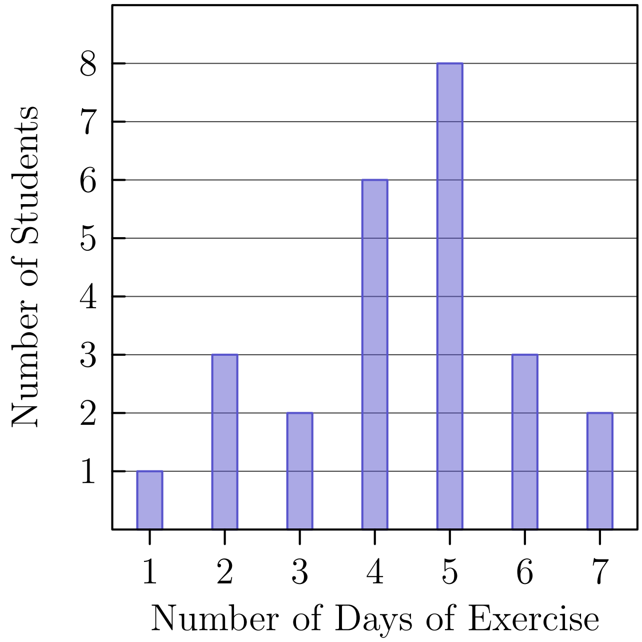
$$2+0+1+8+U=11+U.$$

As $\underline{2}\ \underline{0}\ \underline{1}\ \underline{8}\ \underline{U}$ is divisble by 9, $11+U$ must also be divisible by 9. Also, as U is a digit, we know that inuitively: $0 \le U \le 9$. This means that U can only be 7.

Now we know that the 5-digit number in question is 20187, and we want to find the remainder when we divide 20187 by 8. To solve this, simply use long division to see that $20187 = 2523 \cdot 8 + 3$. Therefore, the remainder is 3.

Thus, the correct answer is **B**.

8. Mr. Garcia asked the members of his health class how many days last week they exercised for at least 30 minutes. The results are summarized in the following bar graph, where the heights of the bars represent the number of students.



What was the mean number of days of exercise last week, rounded to the nearest hundredth, reported by the students in Mr. Garcia's class?

A 3.50

B 3.57

C 4.36

D 4.50

E 5.00

Solution(s):

Counting the number of each occurence, we can see that there are 1 1s, 3 2s, 2 3s, 6 4s, 8 5s, 3 6s, and 2 7s.

Therefore, there are $1 + 3 + 2 + 6 + 8 + 3 + 2 = 25$ students in total.

Therefore, the total number of days of exercise is

$$\begin{aligned} &1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 6 \cdot 4 \\ &+ 8 \cdot 5 + 3 \cdot 6 + 2 \cdot 7 \\ &= 109 \end{aligned}$$

The mean number of days of exercise is defined as being equal to the total number of days of exercise divides by the total number of students, where: \dfrac{\text{total days}}{\text{# students}} = \dfrac{109}{25} \approx 4.36

Thus, **C** is the correct answer.

9. Tyler is tiling the floor of his 12 foot by 16 foot living room. He plans to place one-foot by one-foot square tiles to form a border along the edges of the room and to fill in the rest of the floor with two-foot by two-foot square tiles. How many tiles will he use?

A 48

B 87

C 91

D 96

E 120

Solution(s):

Note that each square foot of the border would require one tile, meaning that the border will take $16 + 12 + 16 + 12 = 56$ tiles. However, notice that this will cause overlapping tiles in each of the four corners, so to fix this, we subtract 4. Therefore, the border will take $56 - 4 = 52$ 1×1 square tiles to completely tile.

Since we have removed one foot from each side due to the border, the remaining rectangle is $10 \text{ feet} \times 14 \text{ feet}$. This must be tiled completely by 2×2 tiles, so it will take

$$\frac{10 \cdot 14}{2 \cdot 2} = 35$$

tiles in total to tile this area.

As it takes 52 1×1 square tiles to tile the border, and 35 2×2 square tiles to tile the remaining area, it will take $52 + 35 = 87$ tiles in total to fill in Tyler's entire living room floor.

Thus, the correct answer is **B**.

10. The *harmonic mean* of a set of non-zero numbers is the reciprocal of the average of the reciprocals of the numbers. What is the harmonic mean of 1, 2, and 4?

A $\frac{3}{7}$

B $\frac{7}{12}$

C $\frac{12}{7}$

D $\frac{7}{4}$

E $\frac{7}{3}$

Solution(s):

The reciprocals of $1, 2,$ and 4 are

$$\frac{1}{1}, \frac{1}{2}, \text{ and } \frac{1}{4}$$

respectively. The average of these reciprocals is

$$\begin{aligned}\frac{(1+\frac{1}{2}+\frac{1}{4})}{3} &= \frac{(\frac{7}{4})}{3} \\ &= \frac{7}{12}.\end{aligned}$$

As the harmonic mean is the reciprocal of the average of the reciprocals of the numbers (which we just calculated to be $\frac{7}{12}$), we conclude that the harmonic mean is $\frac{12}{7}$.

Thus, the correct answer is **C**.

11. Abby, Bridget, and four of their classmates will be seated in two rows of three for a group picture, as shown. \begin{array}[ccc] \text{\text{X}}&\text{X}&\text{X} \\ \text{X}&\text{X}&\text{X} \end{array} If the seating positions are assigned randomly, what is the probability that Abby and Bridget are adjacent to each other in the same row or the same column?

A $\frac{1}{3}$

B $\frac{2}{5}$

C $\frac{7}{15}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution(s):

We can split the problem into two cases. In case 1, Abby is in one of the middle two seats, and in case 2, she is in one of the outer 4 seats.

Firstly notice that there is a $\frac{2}{6} = \frac{1}{3}$ probability of case 1 being true (i.e. Abby is in the middle two seats). For Bridget to be adjacent to Abby in this case, she must be in either of the two seats on the left or the two seats on the right of Abby, or she is in the same column as her. There are 3 ways to make this happen out of a possible 5 open seats, so there is a $\frac{3}{5}$ chance of this happening. Therefore, the total probability of this case is

$$\frac{1}{3} \cdot \frac{3}{5} = \frac{3}{15}.$$

Next, notice that there is a $\frac{4}{6} = \frac{2}{3}$ probability of case 2 being true (i.e. Abby is in the outer four sets). For Bridget to be adjacent to Abby in this case, she must either be in the single seat next (to the left or right, depending on Abby's position), or she is in the same column as Abby. There are 2 ways to make this happen out of a possible 5 open seats, so there is a $\frac{2}{5}$ chance of this happening.

Therefore, the total probability of this case is

$$\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}.$$

Therefore, the final probability of *either* of these cases happening is $\frac{3}{15} + \frac{4}{15} = \frac{7}{15}$.

Thus, **C** is the correct answer.

12. The clock in Sri's car, which is not accurate, gains time at a constant rate. One day as he begins shopping he notes that his car clock and his watch (which is accurate) both say 12:00 noon. When he is done shopping, his watch says 12:30 and his car clock says 12:35. Later that day, Sri loses his watch. He looks at his car clock and it says 7:00. What is the actual time?

A $5{:}50$

B $6{:}00$

C $6{:}30$

D $6{:}55$

E $8{:}10$

Solution(s):

Starting from $12{:}00$ noon, after 30 minutes of time elapsed, the car clock went 35 minutes ahead.

Therefore, for every minute the car clock goes ahead, $\frac{30}{35} = \frac{6}{7}$ minutes of actual time pass by. From the time $12{:}00$ to $7{:}00$, the car goes ahead $7 \cdot 60 = 420$ minutes, and therefore, $420 \cdot \frac{6}{7} = 360$ minutes, or 6 hours, of actual time have passed by. If we start at $12{:}00$ and 6 hours pass by, the time is $6{:}00$.

Thus, **B** is the correct answer.

13. Laila took five math tests, each worth a maximum of 100 points. Laila's score on each test was an integer between 0 and 100, inclusive. Laila received the same score on the first four tests, and she received a higher score on the last test. Her average score on the five tests was 82. How many values are possible for Laila's score on the last test?

A 4

B 5

C 9

D 10

E 18

Solution(s):

Since the average score on the five tests is 82, the total score of those five tests must be $5 \cdot 82 = 410$.

Now, let f be the score on the first 4 tests and let l be the score for the last test.

We know that

$$f < l < 100$$

and

$$4f + l = 410.$$

And as $410 = 4f + l < 5l$, we know $\frac{410}{5} = 82 < l$.

Also, since $4f + l = 410$, and dividing 410 by 4 gives us a remainder of 2, we know that dividing l by 4 must leave a remainder of 2 as $4f$ will leave no remainder when divided by 4. Equivalently:

$$l \equiv 2 \mod 4.$$

Since $82 < l < 100$ and $l \equiv 2 \mod 4$, the only options for l are $86, 90, 94, 98$. This yields four distinct solutions as follows:

$$\begin{aligned}(f, l) = &(81, 86);\\ &(80, 90);\\ &(79, 94);\\ &(78, 98)\end{aligned}$$

Therefore, there are 4 solutions, and **A** is the correct answer.

14. Let N be the greatest five-digit number whose digits have a product of 120. What is the sum of the digits of N?

A 15

B 16

C 17

D 18

E 20

Solution(s):

To make the largest possible 5 digit number, we must maximize the first digit (the digit in the ten-thousands place).

The largest number that is strictly less than 10 and divides 120 is 8, so the first digit must be 8. Therefore, the product of the remaining number is 15.

Similarly, we must now maximize the second digit.

The largest number that is less than 10 and divides 15 is 5, so the second digit is 5. Therefore, the product of the remaining number is 3.

We must then maximize the third digit.

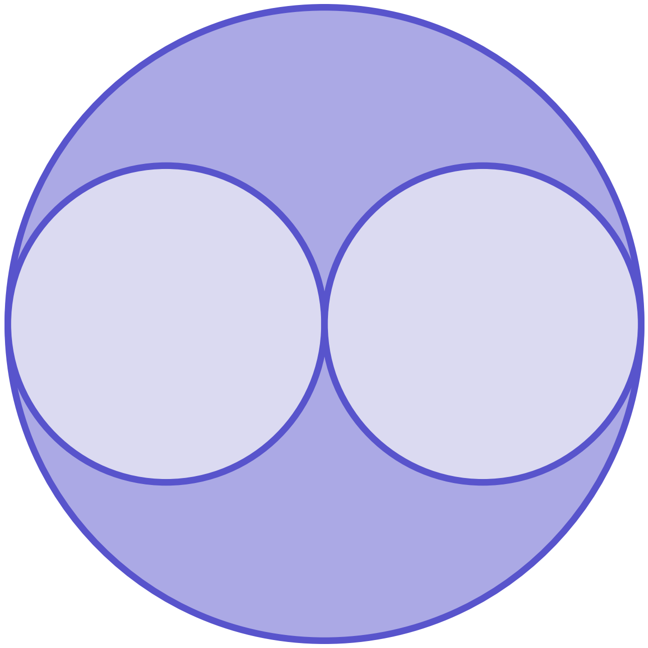
The largest number that is less than 10 and divides 3 is 3, so the third digit is 3. Therefore, the product of the remaining number is 1. This means the 4th and 5th digits are 1.

This makes $N = 85311$, so the sum of the digits is

$$8 + 5 + 3 + 1 + 1 = 18$$

Thus, **D** is the correct answer.

15. In the diagram below, a diameter of each of the two smaller circles is a radius of the larger circle. If the two smaller circles have a combined area of 1 square unit, then what is the area of the shaded region, in square units?



A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D 1

E $\frac{\pi}{2}$

Solution(s):

Let A be the area of the large circle.

Since the diameter of each of the two smaller circles is itself the radius of the larger circle, the radius of the smaller circles half that of the larger circle.

Symbolically, if we allow r_l to be the radius of the large circle and r_s to be the radius of each of the smaller circles:

$$r_s = \frac{1}{2} r_l$$

As the area of the larger circle is equal to $A = \pi r_l^2$, the area of the smaller circles are equal to

$$\begin{aligned} \pi r_s^2 &= \pi \left(\frac{1}{2} r_l\right)^2 \\ &= \frac{1}{4}(\pi r_l^2) \\ &= \frac{1}{4}A. \end{aligned}$$

As the area of two of these smaller circles combined is equal to 1 square unit, then it follows that $2 \cdot \frac{1}{4}A = 1$ square unit, implying that $A = 2$ square units.

As the area of the shaded region is equal to the area of the larger circle (A) minus the combined area of the two smaller circles (1), the area of the shaded region is $A - 1 = 2 - 1 = 1$ square unit.

Thus, the correct answer is **D**

16. Professor Chang has nine different language books lined up on a bookshelf: two Arabic, three German, and four Spanish. How many ways are there to arrange the nine books on the shelf keeping the Arabic books together and keeping the Spanish books together?

A 1440

B 2880

C 5760

D $182,440$

E $362,880$

Solution(s):

Since we are keeping the Arabic books together and the Spanish books together, we can look at each of them as the same object (collections of books).

As such, there are 5 objects on the bookshelf: three German books, one collection of Arabic books, and one collection of Spanish books. There are $5!$ ways to order the 5 objects. As we already have the books together, there are $2!$ ways of ordering the Arabic books and $4!$ ways of ordering the Spanish books. Therefore, the total ways to order the books is

$$\begin{aligned} 5! \cdot 4! \cdot 2! &= 120 \cdot 24 \cdot 2 \\ &= 5760 \end{aligned}$$

Thus, the correct answer is **C**.

17. Bella begins to walk from her house toward her friend Ella's house. At the same time, Ella begins to ride her bicycle toward Bella's house. They each maintain a constant speed, and Ella rides 5 times as fast as Bella walks. The distance between their houses is 2 miles, which is $10,560$ feet, and Bella covers $2\frac{1}{2}$ feet with each step. How many steps will Bella take by the time she meets Ella?

A 704

B 845

C 1056

D 1760

E 3520

Solution(s):

Since for every foot Bella walks, Ella walks 5 feet, we know that Bella will walk $\frac{1}{6}$ of the cumulative distance between the two of them, and so she walks

$$\frac{1}{6} \cdot 10560 = 1760$$

feet. Since she walks 2.5 feet per step, she takes

$$\frac{1760}{2.5} = 704$$

steps by the time she meets Ella.

Thus, **A** is the correct answer.

18. How many positive factors does 23,232 have?

A 9

B 12

C 28

D 36

E 42

Solution(s):

Begin by finding the prime factorization of 23232. To do this, we repeatedly factor out the smallest prime factor from the number, a process that terminates when the number is a prime number. This process is outlined below:

$$\begin{aligned} 23232 &= 2 \cdot 11616 \\ &= 2^2 \cdot 5808 \\ &= 2^3 \cdot 2904 \\ &= 2^4 \cdot 1452 \\ &= 2^5 \cdot 726 \\ &= 2^6 \cdot 363 \\ &= 2^6 \cdot 3 \cdot 121 \\ &= 2^6 \cdot 3 \cdot 11^2 \end{aligned}$$

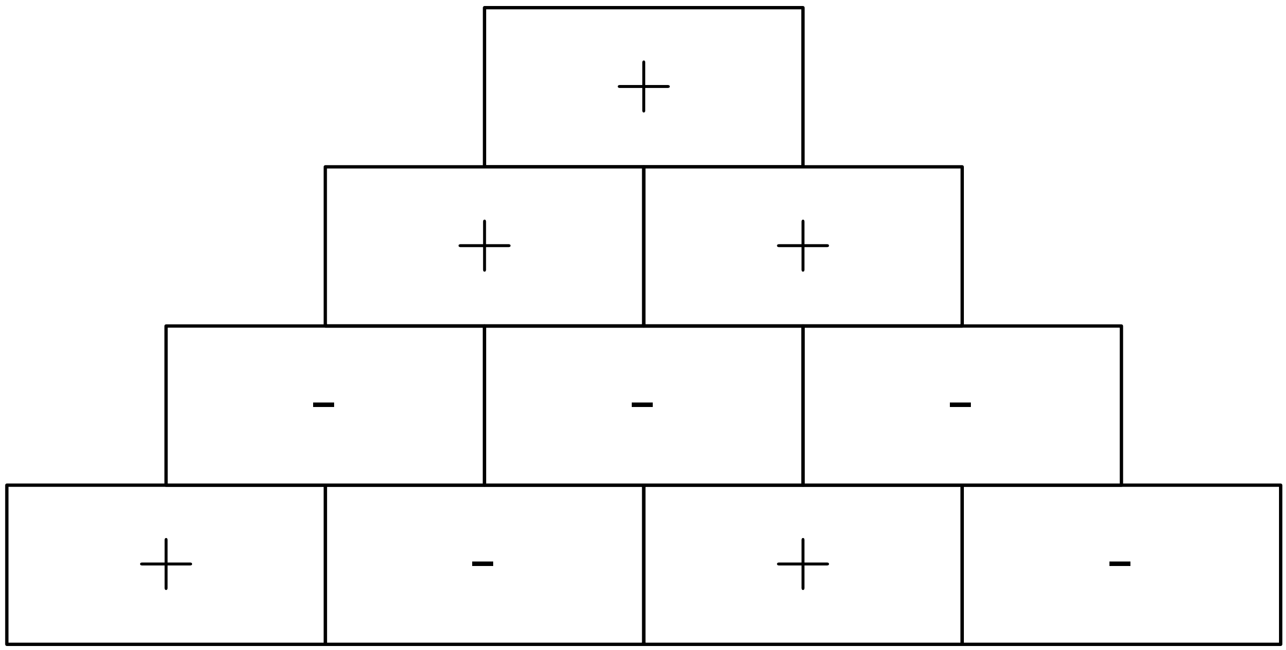
An arbitrary factor of 23232 can be created by taking the product of any number of prime factors. More explicitly, as 23232 can be represented $p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ where p is a prime number, each factor has $(e_1 + 1) \cdots (e_m + 1)$ options of prime factorizations to choose from, and thus, there are $(e_1 + 1) \cdots (e_m + 1)$ factors. Plugging in values, we can see that there are

$$\begin{aligned} (6+1)(1+1)(2+1) &= 7 \cdot 2 \cdot 3 \\ &= 42 \end{aligned}$$

factors of 23232.

Thus, **E** is the correct answer.

19. In a sign pyramid a cell gets a "+" if the two cells below it have the same sign, and it gets a "-" if the two cells below it have different signs. The diagram below illustrates a sign pyramid with four levels. How many possible ways are there to fill the four cells in the bottom row to produce a "+" at the top of the pyramid?



- A 2
- B 4
- C 8
- D 12
- E 16

Solution(s):

Suppose we have two cells and the cell above them. If we are given the bottom left cell and the top cell, we can always find the bottom right cell as follows:

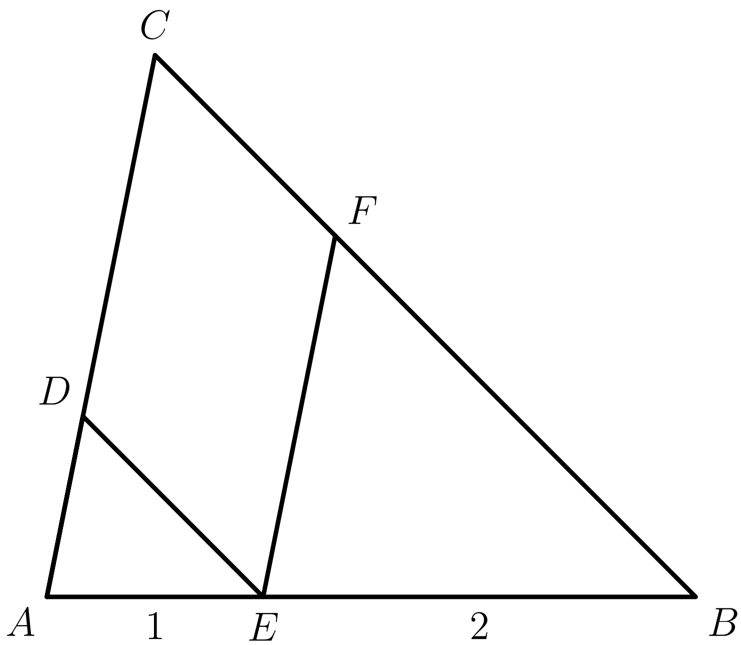
If the top cell is $+$, then the bottom right cell must be the same as the bottom left cell, and if the top cell is $-$, the bottom right cell must be the opposite of the bottom left cell.

Now, suppose we are given a row. Then, suppose we choose a value for the cell below and to the left of the leftmost cell in our given row. We then can inductively determine the entire row below our given by first finding the bottom-right cell of the leftmost cell in our row, and using that newly found cell as the bottom-left reference for the second to the left cell in the given row to find its bottom-right counterpart. The process continues on until the row below the given row is fully solved.

Therefore, since we know that the top row has a cell labelled $+$, we have 2 choices for the row below -- depending on our choice of the bottom-left cell. Similarly, we have 2 choices for the third row, and thus 2 choices for the fourth row. This makes $2 * 2 * 2 = 8$ total choices for the bottom row of the sign pyramid.

Thus, the correct answer is **C**.

20. In $\triangle ABC$, a point E is on $\overline{AB}$ with $AE = 1$ and $EB = 2$. Point D is on $\overline{AC}$ so that $\overline{DE} \parallel \overline{BC}$ and point F is on $\overline{BC}$ so that $\overline{EF} \parallel \overline{AC}$. What is the ratio of the area of $CDEF$ to the area of $\triangle ABC$?



A $\frac{4}{9}$

B $\frac{1}{2}$

C $\frac{5}{9}$

D $\frac{3}{5}$

E $\frac{2}{3}$

Solution(s):

Let the area of $\triangle ABC$ be equal to t. Since $DE \parallel BC$ and $FE \parallel CA$, we can deduce that

$$ADE \sim ABC \text{ and } EFB \sim ABC.$$

Since $AE = \frac{AB}{3}$, the area of ADE is equal to $(\frac{1}{3})^2\ t = \frac{t}{9}$. Since $EB = \frac{AB}{3}$, the area of EFB is equal to $(\frac{2}{3})^2 t = \frac{4}{9}t$. Finally, to find the area of $CDEF$, we take the area of $ABC = t$ and subtract the areas of ADE and EFB. This is equivalent to the expression

$$t - \frac{t}{9} - \frac{4t}{9} = \frac{4t}{9}.$$

Therefore, the ratio of the area of $CDEF$ and ABC is $\frac{(\frac{4a}{9})}{a} = \frac{4}{9}$.

Thus, **A** is the correct answer.

21. How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11?

A 1

B 2

C 3

D 4

E 5

Solution(s):

Suppose x is a number that satisfies these conditions. We know that $100 \leq x \leq 999$.

The first statement implies that

$$\begin{aligned} x &\equiv 2 \mod 6 \\ &\equiv -4 \mod 6 \end{aligned}$$

This, in turn, implies that

$$x + 4 \equiv 0 \mod 6.$$

Similarly, the second statement implies that

$$\begin{aligned} x &\equiv 5 \mod 9 \\ &\equiv -4 \mod 9 \end{aligned}$$

This, in turn, implies that

$$x + 4 \equiv 0 \mod 9.$$

Finally, the third statement implies that

$$\begin{aligned} x &\equiv 7 \mod 11 \\ &\equiv -4 \mod 11 \end{aligned}$$

This, in turn, implies that

$$x + 4 \equiv 0 \quad \mathrm{mod}\ 11.$$

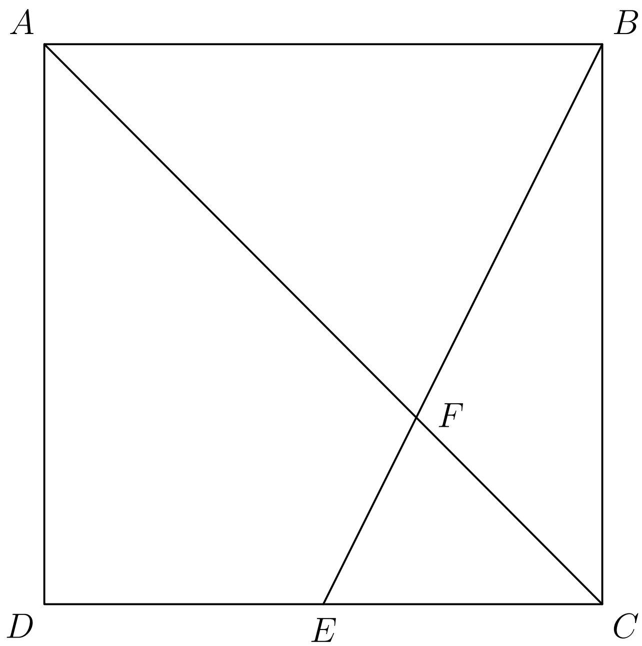
Together, these three conditions mean that $x+4\,|\,6, x+4\,|\,9, x+4\,|\,11$ we know that

$$x + 4\,|\,\mathrm{lcm}(6, 9, 11).$$

Therefore, $x+4\,|\,198$. We also know $104 \le x+4 \le 1003$, so we can see that there are 5 possible values in this interval such that $x+4\,|\,198$.

Thus, **E** is the correct answer.

22. Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$?



A 100

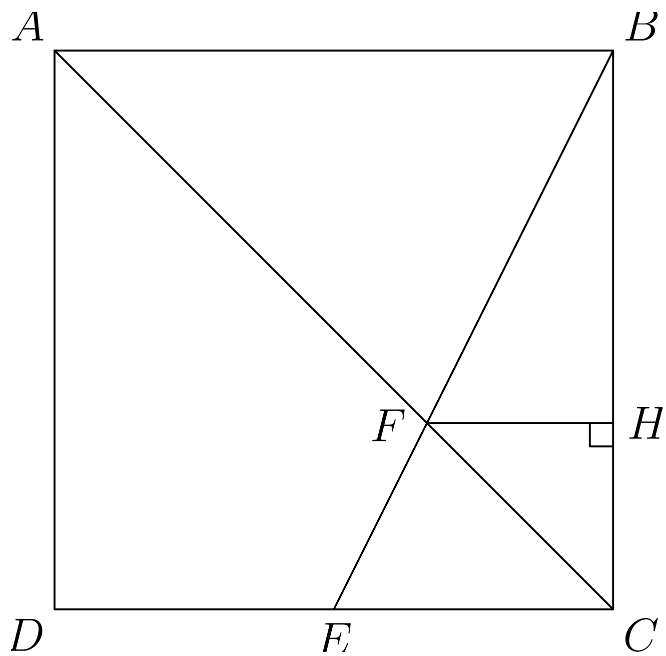
B 108

C 120

D 135

E 144

Solution(s):



Let H be the point on $\overline{BC}$ where the altitude from F to $\overline{BC}$ meets $\overline{BC}$. This altitude, $\overline{FH}$ is illustrated above. Then, by angle-angle similarity, we can see that

$$\triangle CAB \sim \triangle CFH$$

and

$$\triangle BFH \sim \triangle BEC.$$

Since the sides of similar triangles are proportional, we know that

$$\frac{FH}{BH} = \frac{EC}{BC}$$

and

$$\frac{FH}{HC} = \frac{AB}{BC}.$$

Thus,

$$\frac{FH}{EC} = \frac{BH}{BC}$$

and

$$\frac{FH}{AB} = \frac{HC}{BC}.$$

Adding these equations yields:

$$\begin{aligned}\frac{FH}{EC}+\frac{FH}{AB}&=\frac{BH}{BC}+\frac{HC}{BC}\\&=\frac{BH+HC}{BC}\\&=\frac{BC}{BC}\\&=1.\end{aligned}$$

This, in turn, goes to show that

$$\frac{1}{EC}+\frac{1}{AB}=\frac{1}{FH}.$$

Now, let s be the side length of the square. We know $AB = 2 \cdot EC = s$. This means

$$\begin{aligned}\frac{1}{FH}&=\frac{1}{EC}+\frac{1}{AB}\\&=\frac{1}{a}+\frac{2}{a}\\&=\frac{3}{a}.\end{aligned}$$

Therefore, $FH = \frac{s}{3}$.

Now, to compute the area of $\triangle EFC$, we take the area of $\triangle BCE$ and subtract the area of $\triangle BFC$. This is equal to

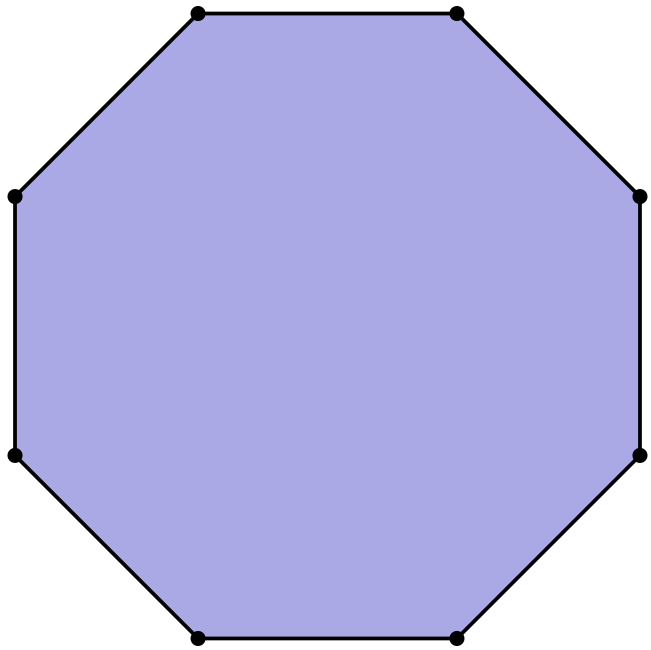
$$\begin{aligned}\frac{BC\cdot EC}{2}-\frac{BC\cdot FH}{2}&=\frac{BC\cdot(EC-FH)}{2}\\&=\frac{s\cdot(\frac{s}{2}-\frac{s}{3})}{2}\\&=\frac{s\cdot\frac{s}{6}}{2}\\&=\frac{s^2}{12}.\end{aligned}$$

The area of $AFED$ is the area of $\triangle ACD$ minus the area of $\triangle EFC$, which is equal to

$$\begin{aligned}\frac{s^2}{2}-\frac{s^2}{12}&=\frac{5s^2}{12}\\&=45.\end{aligned}$$

With $\frac{5}{12}s^2 = 45$, we get $s^2 = 108$, which is the area of the full square.

23. From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?



A $\frac{2}{7}$

B $\frac{5}{42}$

C $\frac{11}{14}$

D $\frac{5}{7}$

E $\frac{6}{7}$

Solution(s):

Without loss of generality, allow A to be a vertex of the triangle. Suppose we also have points B, C of the triangle with A, B, C being in clockwise order. Let x be the number of points (i.e. vertices of the octagon) in between the points of A and B, y be the number of points in between the points of B and C, and z be the number of points in between the points of B and C. We know $x+y+z=5$ as it encompasses every vertex of the octagon except A, B, C.

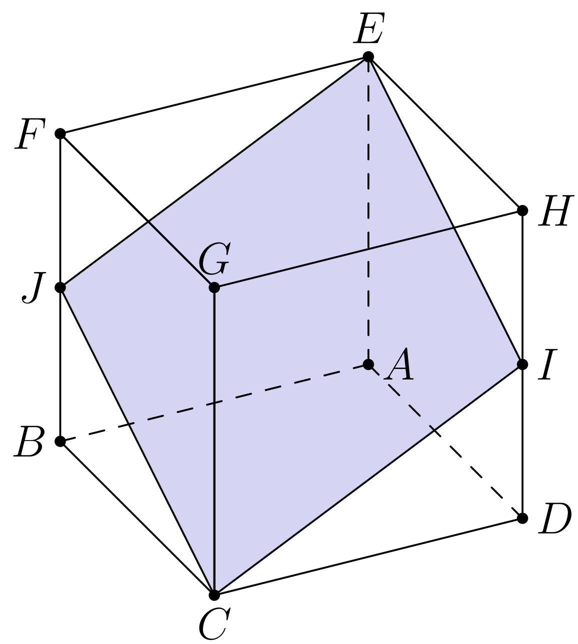
If two sides form the sides of an octagon, the distance between them would be 0.

Therefore, if we use complementary counting to find how many have $x, y, z > 0$, we can deduce out how many triangles are formed with no sides of the triangle being a side of the octagon. This would make x, y, z whole numbers whose sum is 5. Using the stars and bars method, we can see that there are $\binom{5-1}{3-1} = 6$ ways to place B, C such that $x, y, z > 0$. Now to find the total number of cases, since there are 7 points that aren't A, there are $\binom{7}{2} = 21$ ways to place B, C such that they run counter clockwise.

This means there is a $\frac{6}{21} = \frac{2}{7}$ probability of the triangle not having sides on the octagon. Therefore, there is a $1 - \frac{2}{7} = \frac{5}{7}$ probability of the triangle having at least one side on the octagon.

Thus, **D** is the correct answer.

24. In the cube $ABCDEFGH$ with opposite vertices C and E, J and I are the midpoints of segments $\overline{FB}$ and $\overline{HD}$, respectively. Let R be the ratio of the area of the cross-section $EJCI$ to the area of one of the faces of the cube. What is R^2?



A $\frac{5}{4}$

B $\frac{4}{3}$

C $\frac{3}{2}$

D $\frac{25}{16}$

E $\frac{9}{4}$

Solution(s):

Allow s to represent the length of an edge of the cube. Noting that each side of the cross section is equal in length, we conclude that $EJCI$ is a rhombus. The area of this rhombus can be calculated as $\frac{1}{2}IJ \cdot CE$, as the area of a rhombus is equal to half the product of its diagonals. Using the Pythagorean Theorem:

$$IJ = FH = s\sqrt{2}.$$

Similarly, using the Pythagorean Theorem again lets us see that:

$$\begin{aligned} CE &= \sqrt{AC^2 + AE^2} \\ &= \sqrt{(s\sqrt{2})^2 + s^2} \\ &= \sqrt{2s^2 + s^2} \\ &= s\sqrt{3} \end{aligned}$$

Therefore,

$$\begin{aligned} R &= \frac{\frac{1}{2}IJ \cdot CE}{s^2} \\ &= \frac{\frac{1}{2}s^2\sqrt{2}\sqrt{3}}{s^2} \\ &= \sqrt{\frac{3}{2}} \end{aligned}$$

Thus, $R^2 = \dfrac{3}{2}$, and the correct answer is **C**.

25. How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?

- A 4
- B 9
- C 10
- D 57
- **E** 58

Solution(s):

Suppose x is any perfect cube in this range.

If $x^3 \leq 2^{18} + 1$, then

$$x^3 = 2^{18} + 1$$

or

$$x^3 \leq 2^{18} = 2^{6^3}$$

$$\implies x \leq 2^6 = 64.$$

If $x^3 = 2^{18} + 1 = 64^3 + 1$, then it follows that

$$64^3 < x^3 < 65^3$$

$$\implies 64 < x < 65$$

This would mean that x isn't an integer. This is a contradiction, so we know

$$x \leq 64.$$

We also know

$$2^8 + 1 = 257 \leq x^3.$$

Now, suppose $257 \leq x^3 < 343$. Then, we know $216 < x^3 < 343$.

This means $6 < x < 7$, which also means that x isn't an integer. This is a contradiction, so $7 \leq x$.

Therefore, all x which satisfy $343 \le x^3 \le (2^6)^3$ must also satisfy

$$7 \le x \le 64.$$

Therefore, the number of possible x's is $64 - 7 + 1 = 58$.

Thus, **E** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc8/2018